ALGEBRAIC CONFINEMENT AND WILSON LOOP OF SU(N) YANG–MILLS THEORY

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The algebraic confinement property of the theory, namely singletness with respect to the group of gauge transformations at infinity, is shown to be precisely that of providing area law in Wilson loop. Specific cases of SU(2) and SU(3) gluodynamics are studied in more detail, where string tension may be found out numerically in terms of the phase transition temperature. Dependence on the type of boundary conditions in use is discussed.

Fairly sophisticated analysis done in previous papers of the author in collaboration with N. A. Sveshnikov [1-3] has elucidated fundamental concept of color confinement as one that only makes sense while speaking about representation space associated with the equilibrium state of the system in consideration. The confinement property so understood, though being truly dynamical, is formulated simply as requirement of singletness of all states constituting this Hilbert space. The most nontrivial issue however is that the confinement group must be rather large as opposed to naïve global group choice. This group is referred to as the group of *gauge transformations at infinity* G_∞ [1] and its singlet projector prohibits not only globally coloured states, but leak of colour to infinity in any fixed direction too. We will keep away in the present report of discussion on peculiar confinement mechanism responsible for the latter property but will focus rather on consequences of such singletness taking it for granted.

First of all let us emphasize that namely this strong requirement makes it possible to carry out explicit evaluation below. In the deconfinement phase the best we can do is to write down perturbative pieces which can hardly reflect the underlain physical picture. Next, we extensively use hereafter so called 3-d Fock–Schwinger gauge [4-6] $\mathbf{x}\mathbf{A}(\mathbf{x},t)=0$ proved very convenient in nonperturbative treatment. This remarkable gauge plays significant role in describing confinement since its residual dynamics symmetry group is precisely G_∞.

To withstand severe infrared divergences one ought to mitigate the theory by an appropriate regularization. In view of that we are working at nonzero temperature,

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such procedure is anyway indispensable as there is no chance to construct trace of the Gibbs factor right away for the system in infinite volume. We exploit mainly the simplest possibility associated with the considered gauge — system enclosed in spherical box with Dirichlet condition on the boundary, which may be symbolically written as

$$\varphi(t, R\hat{\mathbf{x}}) = 0, \tag{1}$$

where φ is any independent field variable and R is radius of the sphere. Keeping in mind the path integral technique, there will also be the periodicity condition for fields in time at $t = \beta$, where β is inverse temperature.

The symmetry under G_∞ action may well be formulated under retained cut–off R. Generators of the group are expressed via the electric field strengths as follows

$$\Phi_R(\sigma) = -\int d\hat{\mathbf{x}}\, R^2\, \mathbf{E}^a(R\hat{\mathbf{x}})\hat{\mathbf{x}}\, \sigma^a(\hat{\mathbf{x}})\,, \tag{2}$$

they have true algebra

$$[\Phi_R(\sigma'),\, \Phi_R(\sigma'')] = -i\, \Phi_R([\sigma', \sigma''])\,, \tag{3}$$

and what is of importance, commute with the cut–off Hamiltonian

$$[H_R, \Phi_R(\sigma)] = 0, \qquad H_R = \int_{|\mathbf{x}|<R} d\mathbf{x}\; h(\mathbf{x})\,. \tag{4}$$

Here parameters of the group $\sigma(\hat{\mathbf{x}})$ are arbitrary functions of direction in $\mathbf{R}^3$.

Coming closer to our subject, we accept *confinement condition* valid for the temperatures below critical in the following form [1]. Any thermal average

$$\langle A\rangle_{\beta\, R} = Z^{-1}_{\beta\, R}\ \mathrm{Tr}(e^{-\beta\ H_R}\ P_S\ A) \tag{5}$$

besides usual Gibbs factor has to be supplemented by the singlet projector P_S inside it, that is

$$P_S = \int_{G_\infty} d\sigma\ e^{i\Phi_R(\sigma)}\,, \qquad d\sigma = \prod_{\hat{\mathbf{x}}} d\sigma(\hat{\mathbf{x}}) \tag{6}$$

the integral over the Haar measure.

Evaluation of the Wilson loop [7,8]

$$\mathcal{W}[\Gamma] = \langle \mathrm{Tr}\,\mathrm{P}\exp(g \oint_\Gamma A_\mu dx^\mu)\rangle_\beta \tag{7}$$

runs along the following lines. Let us choose extremely convenient rectangular contour of the form drawn on Fig. 1.

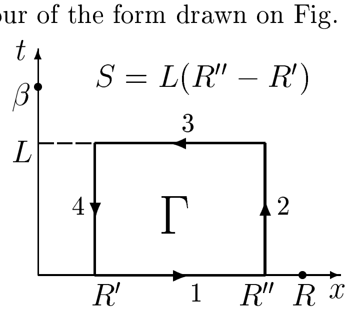

Fig. 1 Integration contour for Wilson loop

Then the integral over sites 1 and 3 is vanishing by the gauge condition. Temporal component of the gauge field in FS gauge is expressed through x-longitudinal part of the electric strength via

$$A_0(t,\,\mathbf{x}) = \int_x^{x_0} dy\;\, E_{\|}(t, y\hat{\mathbf{x}})\,. \tag{8}$$

Putting these two facts together, one gets simply that

$$\oint_\Gamma A_\mu\, dx^\mu \, = -\int_0^L dt\;\, \int_{R'}^{R''} dx\;\, E_{\|}(t,\,\mathbf{x})\,. \tag{9}$$

The latter formula can be rewritten through physical variables $\mathbf{A}_\perp,\ \mathbf{E}_\perp$ of this gauge using exact solubility of the Gauss constraint [1,2]

$$E_{\|}(t,\,\mathbf{x}) = -\frac{1}{x^2}\int_0^x y^2\, dy\;\, \Phi_\perp(t,\,y\hat{\mathbf{x}}), \qquad \Phi_\perp = \nabla \mathbf{E}_\perp\,. \tag{10}$$

Proceeding quite strightforwardly one is able to write down the average of Wilson loop (7) as path integral over canonical transversal variables, which we have to skip here for sake of brevity (see ref. [4].) Singlet projector $P_s$in such a formula may be taken into account by functional delta–function

$$P_s\ \propto\ \prod_{a,\,\hat{\mathbf{x}}} \delta\big(\Phi^a_R\,(\hat{\mathbf{x}})\big) = \prod_{a,\,\hat{\mathbf{x}}} \delta\bigg(\int_0^R x^2 dx\;\, \Phi^a_\perp(\mathbf{x})\bigg)\,. \tag{11}$$

To perform Gibbs averaging some more elaborated technique is needed. In [1,2] there was proposed the method of the effective action denoted as W that is depending on collective variables $\lambda,\ \nu$. One may connect the generating functionals of longitudinal strengths

$$Z[\zeta,\eta] = \int \mathcal{D}\mathbf{A}_\perp\,\mathcal{D}\mathbf{E}_\perp\;\,\exp\,(-)\int d^4x\,[\,\mathbf{E}\dot{\mathbf{A}}_\perp - \frac{1}{2}\mathbf{E}^2 + \frac{1}{2}\mathbf{B}^2 + i\,\zeta E_{\|} - \eta B_{\|}\,] \tag{12}$$

and of collective fields[†]

$$\mathcal{Z}[\zeta,\eta] = \int \mathcal{D}\,\lambda\;\mathcal{D}\,\nu\;\,\exp\;[\,-W[\lambda,\nu] - i\int d^4x\,(\,\lambda\,\zeta + \eta\,\nu\,)\,] \tag{13}$$

via the relation

$$Z[\zeta,\eta] = \exp\,[\,\frac{1}{2}\int d^4x\;\,(\,\zeta^2+\eta^2\,)\,]\;\,\mathcal{Z}[\zeta,\eta]\,. \tag{14}$$

Let $F(E_{\|},B_{\|})$ be an arbitrary functional of its arguments. Then its average

$$\langle\,F(E_{\|},B_{\|})\,\rangle = Z^{-1}[\zeta,\eta]\,F(\,i\,\frac{\delta}{\delta\,\zeta}\,,\,\frac{\delta}{\delta\,\eta}\,)\;Z[\zeta,\eta]\bigg|_{\zeta=\eta=0} \tag{15}$$

is expressed through the $\langle F(\lambda,\nu)\rangle$ of $\mathcal{Z}[\zeta,\eta]$'s by means of the formula

$$\begin{array}{c}\langle\,F(\,E_{\|}\,,\,B_{\|}\,)\,\rangle = \int\mathcal{D}\,\xi\;\mathcal{D}\,\theta\;\,\exp\,(\,-\frac{1}{2}\int d^4x\,[\,\xi^2+\theta^2\,]\,)\\ \langle\,F(\,\lambda + i\,\xi\,,\,-i\,\nu+\theta\,)\,\rangle\,.\end{array} \tag{16}$$

[†]Let us remind that λ is related with the previously used variable σ as follows $\lambda = -\partial_x\sigma$.

Here integrations must be carried out over fields from L^2-space. This relation lies in a heart of collective variables interpretation. They ought be imagined as longitudinal (chromo) electric and magnetic fields smoothed by a Gaussian noise.

Performing this change of integration variables we infer the basic relation

$$\begin{aligned} \mathcal{W}[\Gamma] &= \int \mathcal{D}\xi\, \mathcal{D}\lambda\, \mathcal{D}\nu\; e^{-W[\lambda,\,\nu]-\frac{1}{2}\int_0^\beta dt\ \int_{|\mathbf{x}|<R} d\mathbf{x}\ \xi^2} \\ &\quad \mathrm{tr\ T\text{-}exp}\Big(ig \int_0^L dt\ \int_{R'}^{R''} dx\ [\lambda + i\xi](t,\,\mathbf{x}) \Big)\ P_s[\lambda]\,. \end{aligned} \tag{17}$$

As we have already mentioned, Dirichlet boundary condition (1) and periodicity in time are assumed for all fields. Foregoing variables have simple physical meaning — λ and ν are Fourier – conjugate to longitudinal components of electric $E_\parallel$ and magnetic fields $B_\parallel$ correspondingly. Functional $W[\lambda,\,\nu]$ is referred to as *effective action* and was evaluated in the cited work. Variable ξ is just an auxiliary field appearing under variables change and accounts for fluctuations of electric field.

Now introduce complete and orthonormal basis in the functional space over which integration is performed

$$\begin{gathered} f^k_{n,\,lm} = \left(\tfrac{2\pi}{R}\right)^{1/2}\ e^{i\omega_n t}\ \tfrac{\sin(\Omega_k x)}{x}\ Y_{lm}(\hat{\mathbf{x}})\,, \\ \omega_n = \tfrac{2\pi}{\beta} n\,,\quad n \in Z\,; \qquad \Omega_k = \tfrac{\pi}{R} k\,,\quad k = 1,\ 2, \ldots \end{gathered} \tag{18}$$

If φ is connected with λ in the same way as $\Phi_\perp$ with $E_\parallel$, namely

$$\lambda(\mathbf{x}) = -\frac{1}{x^2} \int_0^x y^2 dy\ \varphi(y\hat{\mathbf{x}})\,, \tag{19}$$

then in terms of Fourier coefficients of φ in the basis (18), $P_s[\lambda]$ will acquire especially simple form

$$P_s\ \propto\ \prod_{n,\,lm} \delta\Big(\sum_{k=1}^{\infty} \frac{(-1)^k}{\Omega_k}\ \varphi^k_{n,\,lm} \Big)\,. \tag{20}$$

Parameters of the contour $R',\ R''$ are at our disposal. The trick is to choose them close to the cut – off R. By translation invariance of $\mathcal{W}[\Gamma]$, if $R',\ R'' < R$ this may actually be done. In the limit $R \to \infty$ there will be no doubts left in artifacts due to the boundary or something like that. So we imply the following conditions

$$R_1\ \equiv\ R - R' > 0\,,\quad R_2\ \equiv\ R - R'' > 0\,,\quad R_i/R\ \ll\ 1\,,\quad i = 1,\ 2 \tag{21}$$

valid hereafter.

One can see from the formula (17) that the auxiliary variable ξ is only mixed up with the physical variables via the term

$$I_{n,\,lm} = \Big(\int_{R'}^{R''} dx\ \lambda(\mathbf{x}) \Big)_{n,\,lm} = \left(\frac{2}{R}\right)^{1/2} \sum_{k=1}^{\infty} \frac{\varphi^k_{n,\,lm}}{\Omega_k^2}\ \left(\frac{\sin(\Omega_k R'')}{R''} - \frac{\sin(\Omega_k R')}{R'} \right)\,. \tag{22}$$

By virtue of the boundary condition (1) and condition (21) it can be rewritten as

$$I_{n,\,lm} = -\left(\frac{R_2}{R''} - \frac{R_1}{R'} \right)\ \left(\frac{2}{R}\right)^{1/2} \sum_{k=1}^{\infty} \frac{(-1)^k}{\Omega_k}\ \varphi^k_{n,\,lm} + O\left(\left(\frac{R_i}{R}\right)^3 \right)\,,\quad R \to \infty\,. \tag{23}$$

Hence by the presence of singlet projector (20) the first term vanishes identically and, consequently, mixing term $I_{n,lm} \to 0$ as R approaches infinity.

This circumstance enables wonderful factorization property of Wilson loop. Extremely sophisticated quantum dynamics embodied inside effective action factorizes away from (17) onto just a normalization constant. Therefore we have to evaluate very simple integral over auxiliary field ξ

$$\mathcal{W}[\Gamma] = \int \mathcal{D}\xi \; e^{-\frac{1}{2}\int_0^\beta dt \; \int_{|\mathbf{x}| \le R} d\mathbf{x} \; \xi^2} \; \mathrm{tr} \; e^{-g \int_0^L dt \; \int_{R'}^{R''} dx \; \xi(t, \mathbf{x})} \, . \tag{24}$$

To get the previous formula we have also used extra gauge transformation to cancel time dependence in $\hat{\xi} = \xi/\xi$. Possibility of the latter was proved in [4], so T-exponential reduces to a usual one. To get rid of colour structure one should again rotate $\hat{\xi}$ by global transformation this time so that it will lie in Cartan subalgebra.

After these steps and performing of Gaussian integration Wilson loop takes finally the form

$$\mathcal{W}[\Gamma] = \int \prod_{\alpha=1}^{N-1} \mathcal{D}\xi^\alpha \, e^{-\frac{1}{2}\int d^4x \, (\xi^\alpha)^2} \sum_{i=1}^{N} \exp\left[\sum_{\alpha=1}^{N-1} (j \bullet \xi^\alpha) \, \lambda_i^{(\alpha)}\right] \, , \tag{25}$$

$$j(x) = \frac{i\,g}{2}\,\theta(0 \le t \le L)\,\frac{\theta(R' \le x \le R'')}{x^2}\,\delta(\hat{\mathbf{x}} - \hat{\mathbf{x}}') \, , \tag{26}$$

giving rise to the result

$$\mathcal{W}[\Gamma] = \sum_{k=1}^{N} \exp[-\sum_{\alpha=1}^{N-1} (\lambda_i^{(\alpha)})^2 \, \chi_0 \, S] \, , \tag{27}$$

where $\left\{\lambda_i^{(\alpha)} \equiv (\lambda^\alpha)_{i\,i} \, , \; (\alpha = 1, \ldots, \; N-1)\right\}$ are elements of diagonal generators of fundamental representation of SU(N).

Lambda–matrices are normalized as

$$\mathrm{tr}(\lambda^a \lambda^b) = 2\delta^{ab} \, , \tag{28}$$

then the factor is equal to

$$\sum_{\alpha=1}^{N-1} (\lambda_i^{(\alpha)})^2 = 2\left(1 - \frac{1}{N}\right) \tag{29}$$

Combining of the previous results gives rise to the area law

$$\mathcal{W}_{SU(N)}[\Gamma] = N\, e^{-\chi_{SU(N)}\, S} \, , \tag{30}$$

with the string tension

$$\chi_{SU(N)} = 2\left(1 - \frac{1}{N}\right)\chi_0 \, . \tag{31}$$

The constant χ_0 contains either ultraviolet and infrared divergences:

$$\chi_0 = \frac{g^2\,\delta(\hat{0})}{8\,R'\,R''} \, , \qquad \delta(\hat{0}) = \sum_l \frac{(2\,l+1)}{4\,\pi} \, . \tag{32}$$

The former is given by angular delta–function with coinciding arguments.

In fact, the final result alone would have no significance without renormalization procedure. Fortunately, there exists another pattern comparison with which may bring forth some useful information. Really, evaluation of free energy density carried out in the previous paper [1,2] is perfectly generalizable on the case of Dirichlet boundary condition.

It turns out that dependence of this function on the same type of regulators is completely combined in dependence on precisely the combination (32):

$$\mathcal{F}_R = \frac{6\,\pi}{g^2\,\beta^2}\,\chi_0\,F_R\left(\frac{4\,\pi^3}{3\,\beta^2\,\chi_0}\right). \tag{33}$$

On the other hand confinement phase transition occurs when $F_R(a)$ acquires extra minima at certain value a_c. Since this critical value is calculable analytically [1,3] one is able to determine dimensionless ratio

$$\xi \;\equiv\; T_c/\sqrt{\chi} \tag{34}$$

for different groups SU(N). In the case $N = 2$ group factor drops out and the value $\xi_0 = T_c^{SU(2)}/\sqrt{\chi_0}$ will be a fundamental constant. But for particular regularization used certain doubt persists whether such quantity is really universal. In order to answer this question we study two extremes — sphere with the radius R and cube with the site $2R$. In the latter case it is naturally to choose periodic boundary condition. Generalization of formula (33) valid for these two cases is

$$\begin{aligned} \mathcal{F}_R = \gamma_{_R}\,F_R[a_{_R}]\,, \qquad \gamma_{_R} &= \tfrac{\delta(\hat{0})}{\beta^2}\,\tfrac{8\,\pi^2\,R}{V_R}\,, \\ a_{_R} &= \tfrac{V_R}{8\,\pi^2\,R}\,\tfrac{(2\pi)^4\,\beta^{-2}}{2\,g^2\,\delta(\hat{0})}\,, \end{aligned} \tag{35}$$

where F_R approaches the same limiting function F as R tends to infinity. Since volumes in both cases are $\frac{4\pi}{3}\,R^3$ and $8\,R^3$, the ratios $\xi_0^{sphere} = \frac{\sqrt{a_c}}{\pi}\sqrt{\frac{6}{\pi}}$ and $\xi_0^{cube} = \frac{\sqrt{a_c}}{\pi}$ differ by the factor about $\sqrt{2}$. Yet we possess no knowledge about how much this slight difference physical or is it on the contrary just artifact of the approximations involved. From now on we examine thermodynamic limit obtained from the case of cubical system. For the time being free energy density was found for particular cases of SU(2) and SU(3) groups only.

Free energy density (effective potential) in SU(2) case undergoes confinement phase transition at a_c [1,3]

$$a_c = 2\,\sqrt{2} - 8\,\sum_{k=0}^{\infty}\,\frac{(4k+1)!!}{(2k+1)!\;2^{2k+1}}\;[\big(1 - \frac{1}{2^{4k+3}}\;\zeta(4k+3) - 1]\;\approx\;2.61882 \tag{36}$$

This transition gives rise to nonzero value of the gauge field temporal component A_0.

For SU(3) the whole picture is a bit more complicated. Nontrivial minima of the free energy density are drawn on Fig. 2 by circles. Those corresponding to light circles $\mathbf{s}_0$ arise at higher temperature $T_c^{(1)}$ and those of dark circles $\mathbf{s}_\pm$ — at

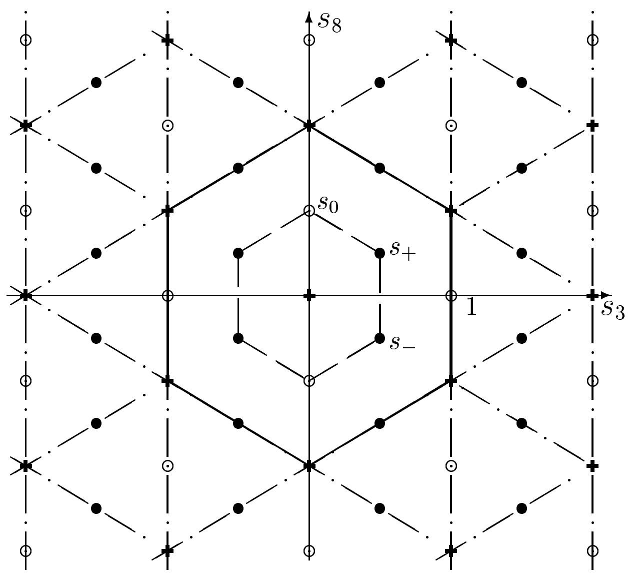

Fig. 2 Minima of the free energy density versus $s = \frac{g\,\beta}{2\pi}\,\sigma$

$T_c^{(2)} = \sqrt{3}/2\ T_c^{(1)}$. The second temperature is in fact the actual temperature of confinement phase transition with

$$\xi_{_{SU(3)}} = \frac{3}{2\sqrt{2}\,\pi}\sqrt{a} \approx 0.55 \tag{37}$$

Analogical value from Monte–Carlo simulations on cubical lattice [11-15] is 0.58 ± 0.04. Another data are summarized in Tab. 1. The principal conclusion to be made from foregoing considerations is in direct relation between algebraic confinement property and linear rising potential of two interacting static quarks as found from Wilson loop. The former is formulated in terms of requirement that all observed states are singlets under the group of *gauge transformations at infinity* G_∞.

Table 1

Comparison of theoretical predictions and lattice data

ratio	lattice	theoretical
$\frac{T_c^{SU(3)}}{T_c^{SU(2)}}$	1.16 ± 0.07	$\sqrt{3/2} \simeq 1.22$
$\frac{\xi^{SU(3)}}{\xi^{SU(2)}}$	1.04 ± 0.18	$3/(2\sqrt{2}) \simeq 1.06$

There is no reference in the definition on some specific gauge condition, it is completely gauge invariant. Although intermediate steps were done using such condition, namely Fock–Schwinger gauge, the best manifestation of self–consistency of our procedure is covariant final result.

It is worthwhile to emphasize how crucial G_∞ group was for obtaining of the area law. Singlet projector P_s disentangles different degrees of freedom in FS gauge, so fluctuations of electric field responsible for the effect under consideration were realized via simply Gaussian integral.

Discovered dependence on the type of the boundary conditions may pose certain problem in phenomenology of confinement phase transition. It demonstrates that at some extent the situation is more complex than it would seem.

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