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Boundary Effects and Confinement in the Theory of Nonabelian Gauge Fields

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Abstract

The thesis is devoted to the problem of colour confinement in the non–Abelian Yang–Mills theory (gluon part of Quantum Chromodynamics). A generalisation of the 3-dimensional Fock–Schwinger gauge is proposed where the Gauss law constraint is exactly solvable. This simplifies the theory in a finite domain and incorporates the variables at the boundary into the Hamiltonian formalism. The dependence of the partition function on the boundary value of the longitudinal component of the electric field is studied and related to the mechanism of the confinement–deconfinement transition. The free energy density is calculated for $SU(2)$ and $SU(3)$ gluodynamics in the mean–field approximation for the collective variables. Analysis of its minima reveals a phase transition at a certain temperature, below which the mean collective variables have nonzero values. This can be interpreted as a confinement–deconfinement phase transition. In the confinement phase the chromo–electric flux through any element of the boundary is strictly zero. This means the singletness with respect to the group of the residual gauge transformations and hence impossibility of observing coloured objects at spatial infinity (in asymptotic states). It is demonstrated that our confinement condition satisfies the traditional confinement criteria. The Wilson loop for $SU(N)$ theory is shown to satisfy the area law. The ratio of the transition temperature to the square root of the string tension coefficient is in a qualitative agreement with the result from lattice Monte Carlo simulations. In the deconfinement phase the global symmetry Z_N (centre of $SU(N)$) is spontaneously broken by the surface terms. The confinement phase is characterised by unbroken symmetry with all nontrivial minima having the same depth and transformable by Z_N actions. (Translated from Russian manuscript, Moscow, 1995).

1 Introduction **2**

2 Some properties of the generalised Fock–Schwinger gauge **12**

3 Hamiltonian formalism of the Yang–Mills theory in a finite domain **18**

4 Yang–Mills theory in the variables at infinity formalism **25**

4.1 Some background from the formalism of the variables at infinity 26

4.2 Dynamics at infinity in the Yang–Mills theory 32

4.3 Poincaré algebra in the Fock–Schwinger gauge 36

5 Dependence of the partition function on the variables at the boundary **37**

5.1 Electrodynamics with an external charge 38

5.2 Formulation in terms of the collective variables 41

5.3 Effective action of gluodynamics 43

6 Confinement criteria. The Wilson loop **53**

7 Conclusion **61**

8 Appendices **64**

1 Introduction

The problem of confinement in the non–Abelian gauge theory has attracted the keenest interest of researchers for several decades and it is one of the most urgent unresolved issues for the justification of the Standard Model of strong and electroweak interactions. So far, all attempts to search for free quarks lead to the unambiguous conclusion that they are not observed in the scattering experiments of the elementary particles at energies up to the order of dozens of GeV [1, 2]. Back in the octet model of quarks, it was assumed that objects with non-zero colour charge are bound by powerful attractive forces that do not allow them to exist in free asymptotic states, and therefore be observed experimentally. In order to truly hold particles in a small region of of space, the interaction potential should be linearly increasing with the separation [3].

Quantum chromodynamics (QCD), based on the theory of the Yang–Mills [4] non–Abelian gauge fields [5, 6] is considered a well–established model for the description of strong interactions for over twenty years. The discovery of the phenomenon of the asymptotic freedom [7, 8, 9] has led to notable advances in the application of the perturbation theory to describe processes at high energies [10, 11]. However, the essential non–perturbative nature of the phenomena at low energies has not yet allowed us to reach a clear understanding and the explanation of a number of problems, among which we should mention the confinement, the spontaneous chiral invariance breaking and the hadron mass spectrum. Despite significant progress in the study of many important aspects of the confinement problem [12, 13], in the scientific community there is still no firm agreement between different approaches that attempt to derive confinement from the fundamental laws of QCD.

Allow me to present a brief overview of these approaches, without attempting to follow a chronological sequence, and only focusing on those that I consider to be of greatest theoretical interest. Of course, the literature on the present issue is extremely extensive, therefore my presentation does not pretend to be exhaustive.

Although the concept of confinement was originally introduced to explain why quarks can only be observed as constituents of hadrons, confinement must explain, first of all, the unobservability of coloured states of quanta of the gauge field – gluons. The most common confinement criterion is based on the behaviour of the the Wilson loop, which characterises the potential energy of the interactions between two trial static sources of colour charge, though confinement of gluons does not follow from the latter. It is reasonable to expect that the essence of the confinement mechanism owes its origin

precisely to the dynamics of gauge fields. Therefore, we restrict ourselves to the consideration of quantum gluodynamics and we shall not touch the question of the inclusion of dynamical quarks at present.

Numerous attempts have been made to study the confinement problem in mathematically simpler theories, which are prototypes of QCD. First of all, confinement exists in gluodynamics in two spatial dimensions where it is, however, fulfilled automatically, since the Coulomb potential of a point charge in two dimensions is linearly growing. Nevertheless, a lot of interesting physical information on the spectrum of states of the theory has been obtained from these studies [14].

The study of compact electrodynamics in three dimensions has been intensively carried out both analytically [15-18] and numerically [19]. This theory has a number of interesting properties, such as confinement, dynamical mass generation, and a non–zero coefficient of string tension. It is remarkable that in this theory the monopoles are instanton solutions and taking into account their contribution already gives the correct value of the Wilson loop. In other words, the condensation of instantons explains the confinement phenomenon in the compact QED_3. The latter can be interpreted as a dual Meissner effect in the theory of superconductivity and is accompanied by the appearance of Abrikosov vortices. In this case, the magnetic flux is trapped in narrow tubes connecting the monopole–antimonopole pairs. Another interesting analogy is with the theory of superfluidity. The confinement phase corresponds to the normal phase of ^4He and has zero dielectric permittivity [17]. The ideas born in the framework of compact QED_3, such as the consideration of monopoles and instantons, have subsequently played a significant role in the search for a confinement mechanism in 4-dimensional chromodynamics.

The first evidence in favour of confinement in QCD was the growth of the the running coupling constant at large distances, found in starting orders of the perturbation theory, and the presence in the theory of severe infrared divergences [20, 21]. A physical picture that can be associated with a linearly growing potential was proposed in Ref. [22]. It was supposed that the electric flux between particles in a quark-antiquark pair is squeezed into a narrow tube (string), the thickness of which in the transverse direction does not depend on the distance between quarks. The idea of the string model in QCD looked attractive also because of the agreement of the of the spectrum of some baryons with the Regge trajectories picture [23].

S. Mandelstam [24, 25] proposed a model of the vacuum as a condensate of gluons and light quark–antiquark pairs by analogy with the theory of superconductivity, where the condensation of electron

pairs leads to the Meissner effect – the exclusion of the magnetic flux from the domain. In non–Abelian theory, monopoles exist as natural classical solutions and do not require their explicit introduction at the level of elementary particles. As an example of a theory in which monopole solutions are known one can cite the Georgi–Glashow model, where the gauge symmetry is broken to an Abelian subgroup by the Higgs mechanism [26]. Based on the ideas of the electric–magnetic duality [27, 28] the classification of phases in the $SU(2)$ gauge theory was given: perturbative; Georgi–Glashow one; phase with complete spontaneous symmetry breaking; confinement phase. In the above mentioned works the evidence was obtained that the variational estimate for the energy of the vacuum containing a gas of monopoles is less than that for the perturbative vacuum. Nevertheless, the monopole approach has encountered a number of serious technical problems and the work in this direction continues both analytically and with the help of numerical methods on the lattice (see for example Refs. [29, 30, 31]).

A closely related approach is based on the circumstance that the QCD vacuum is unstable with respect to the introduction of a nonzero mean value of the magnetic field. G. Savvidi [32, 33] calculated the energy correction in the one–loop approximation and found that it has the form $\Delta E \propto -\mathcal{H}^2 \log \mathcal{H}$. However, immediately there arose a question about the Lorentz invariance of such a vacuum. There were considerable attempts to complicate the structure of the vacuum (periodic structure, "spaghetti", etc.), later called the "Copenhagen" vacuum, and thus guarantee its Lorentz invariance [34-37].

I would like to note the work of Yu.A. Simonov [38], in which it is supposed that the QCD vacuum contains quasiclassical background fields, but instead of explicitly specifying the field configurations some assumptions are made about the bilocal correlators of the background fields at large distances. Then the Fock–Schwinger covariant gauge is used to relate the field strengths to the gauge potential fields. Although this approach is somewhat phenomenological, it reproduces reasonable values of the string tension and the magnitude of the condensates. The approach proposed in this thesis, while different in most technical details, shares one common idea with the former work – in order to understand confinement it is more convenient to work with the effective action of the strength fields, which is particularly straightforward in the Fock–Schwinger gauge (the 3-dimensional version in our case).

It seemed very tempting to study the effect of instantons on the confinement — those are classical solutions of the Yang–Mills theory with a finite Euclidean action. One can consider as the most advanced results in this direction, apparently, those achieved in Refs. [39, 40, 41]. In four dimensions instantons depend on the dimensional parameter λ. Assuming that it may be arbitrary, the theory

suffers from serious infrared divergences. On the other hand, it is shown that the explicit introduction of a cut–off leads to the short–range character of the instantons and their inability to produce the confinement phenomenon. It was proposed to consider the plasma of "merons" — long-range objects, 3-dimensional cross sections of which are monopoles, whereas a pair of merons is topologically equivalent to an instanton. The possibility of explaining the confinement by means of instantons in QCD was questioned in Ref. [42] and apparently, although they do improve the perturbative vacuum, the instantons alone are insufficient for the confinement.

An interesting approach is based on an attempt to apply a non–perturbative Ansatz in the Schwinger–Dyson (Bethe–Salpeter) equations. The fact of the existence of confinement certainly has profound consequences on the asymptotics of the Green's functions and their analytic structure (see e.g. [46, 47, 48] and references therein). The interaction energy of static quarks can be represented as the Fourier image of the temporal component of the gluon propagator

$$E(R) \sim \int d\mathbf{k}\, \exp(-i\mathbf{kR})\, \Delta_{00}(\mathbf{k}^2) \propto R^{\alpha-3}, \quad \Delta_{00}(\mathbf{k}^2) \propto k^{-\alpha},\ k \to 0.$$

The free propagator gives $\alpha = 2$ and hence the Coulomb's law. Confinement requires an infrared singularity of the propagator with $\alpha = 4$. In finite order perturbation theory, the maximum the singularity that can be achieved is $k^2 \log k^2$. The calculations of Ref. [43] were performed in the covariant gauge, and in Refs. [44, 45] in the axial gauge. Despite some differences in the details, they were based on a truncation of the Dyson–Schwinger equation and neglecting some terms. Numerical analyses of the resulting equations gave encouraging results, but there was some disagreement between the closure schemes in the chain of equations in different gauges, so the method can not be considered as systematic.

The ability to work directly with the Wilson loop $W[C]$ is an advantage of the effective loop theories derived from QCD [49]. Their idea is based on obtaining integro–differential equations for $W[C]$ upon changing the shape of the loop parametrized by the coordinates $x_\mu(s)$ as in Refs. [50, 51]. The interpretation in terms of a string is given by an approximate equation of the type

$$\frac{\delta^2 W[C[x(s)]]}{\delta x_\mu(s) \delta x^\mu(s)} = \chi^2 \left(\frac{dx}{ds}\right)^2 W[C] + \ldots$$

Note that the loop approach deals with mathematically ill–defined products of singular operators and is therefore ill–suited for the development of further systematic computational schemes. We note also the alternative loop approach of Refs. [52, 53], which can be very promising in combination with the

$1/N$ decomposition ($N \to \infty,\ g^2 N = const$), introduced in Refs. [54, 55].

As we have already mentioned, the Wilson criterion does not follow from the confinement of gluons in general. A more systematic approach to the phenomenon in question should explain the structure of the state space of the system, which in the confinement phase is a subspace, distinguished by some additional singletness condition. Such a condition cannot simply be the condition of global singletness. The latter does not forbid, for example, a state in which the quark and the antiquark are separated by macroscopically large distances and can therefore be separately observable. An attempt to construct such an algebraic theory of confinement was undertaken by T. Kugo and I. Ojima[56] on the basis of their proposed explicitly covariant operator formulation of the Yang–Mills theory [57, 58] in the framework of the BRST formalism, which is a natural generalisation of the Gupta–Bleuler formalism in QED.

The states of the physical space in their formalism nullify the generators of BRST transformations and scale transformations of the ghosts, which thereby ensures the unitarity of the S-matrix. The confinement mechanism proposed by Kugo and Ojima is called quartet (due to the transformation properties of the asymptotic fields) and their confinement criterion consists of two conditions ensuring the cancellation of the ghost degrees of freedom. The physical meaning of these conditions consists in the requirement of the absence of massless single–particle Nambu–Goldstone mode interacting with the BRST current. Given that this criterion is fulfilled, the coloured asymptotic states are not observable and can exist only in the unphysical sectors of the Hilbert space. A similar formalism can be extended to the theory at non–zero temperature [59], but in some papers there are claims that its "kinematic" character of confinement is trivial and the formalism itself is internally contradictory. For example, in Ref. [60] it is argued that there are no representations of the extended BRST algebra satisfying the Kugo–Ojima conditions. It is known that the quartet mechanism works well in the perturbative Higgs phase [61], but it apparently cannot be considered as a general confinement criterion.

The theory of gauge fields on a lattice [62, 63] has proved to be one of the most powerful tools for studying the confinement problem. First of all, it was found that lattice theory in the strong coupling approximation gives the area law in the Wilson loop, i.e. possesses confinement. The investigation of this approximation at non–zero temperatures showed that the colour is not confined at high temperatures, and that both regimes are separated by a phase transition [16, 64]. The real interest is indeed in the continuum limit of lattice theory, which must be carried out in the neighbourhood of the zero coupling constant. For this reason, the strong coupling approximation does

not allow us to draw conclusions about the continuous theory. Significant progress in the study of lattice theory at weak coupling was obtained based on the Monte Carlo method [65]. It has allowed the authors to estimate the values of the phase transition temperature and the string tension in the continuous limit through the application of the renormalisation group techniques [66-69]. A natural order parameter in the lattice theory is the mean Polyakov line,

$$\mathcal{P}(\mathbf{x}) = \text{tr T-exp}\left(\int_0^\beta dt\, A_0(t,\mathbf{x})\right),$$

and the correlator of two Polyakov lines at different points allows us to determine the correlation length and hence deduce the type of the phase transition [70, 71]. Work in this direction has revealed the important role of the global group Z_N of the centre of $SU(N)$, which was first realised in the approach based on the ideas of duality. There is a simple connection of the confinement–deconfinement phase transition with the general theory of critical phenomena [72], which is accompanied by a spontaneous breaking of Z_N symmetry in the deconfinement phase. It is found that the phase transition is a transition of the second order in $SU(2)$ and of the first order in $SU(3)$ gauge theories [73, 69, 74, 75]. At present moment these results are considered to be firmly established in the lattice theory (see e.g. [76] for an overview of the current situation), but the order of the phase of the transition can change in the continuum limit. The main efforts of the groups presently performing calculations on increasingly powerful supercomputers, are focused on problems of the next order of complexity — inclusion of quarks and the computation of the spectrum of bound states in the theory.

Despite astonishing advances in computational methods, there is still considerable dissatisfaction among theorists. First of all, it is not certain whether the qualitative properties of the phase transition persist in the continuous limit (not to mention quantitative ones, although by applying reasonable renormalisation methods some characteristic quantities have been found to agree with known experimental values). The situation improves markedly with increasing accuracy, nevertheless, the main problem is that numerical values are obtained without understanding the underlying physics of the phenomena, and so far none of the of the proposed theoretical schemes have been able to demonstrate their full agreement with the lattice data, or predict a result that would subsequently find a good confirmation numerically.

For these reasons, novel approaches to the explanation of confinement are being offered yet again. In our opinion, a generic variational approach may turn out to be promising as such. The main difficulty there consists in the proper choice of the trial functional and the the possibility of exact

calculation of the mean values with its weight. In Ref. [77] the trial states are explicitly gauge invariant and reduce to Gaussian ones at zero coupling constant. It is found that the energy minimum is reached at a value of the variational parameter away from the perturbative one. Another possibility [78] is to work in terms of only the physical degrees of freedom and a positively defined Hilbert space, thereby reducing the arbitrariness in the choice of trial functions and giving legitimacy to the to the standard variational inequalities (Bogoliubov–Gibbs, Feynman, etc.).

The work of Ref. [79] emphasises the importance of explicitly resolving the Gaussian constraint in order to describe the confinement. Note that the way of resolving the constraint (in the formalism without fixing a gauge) and the choice of variables in that paper differs from the one used by us here.

As a criticism of some claims about the triviality of confinement (for example, in view of the possibility to work completely in terms of of gauge–invariant objects) [80, 81], and thereby the very absence of the problem, we note that they do not explain at least two facts firmly established on the lattice: the linear interaction potential of a quark–antiquark and the absence of confinement at high temperatures.

To conclude the review, allow me to focus on the mechanism of the so–called A_0-condensation. Although, in our opinion, it has not led to major advances and its very methodology is not quite justified, nevertheless, the main object of its study coincides with the that of our present approach, namely the effective action expressed in terms of the temporal component of the gauge field A_0 is the main descriptor of the confinement–deconfinement phase transition. The main idea of this approach is to explain the non–zero mean value of the Polyakov line in the deconfinement phase by a non–zero mean value of A_0, to which we attribute the role of the Higgs field spontaneously breaking Z_N [82, 83] symmetry. As analytical methods were plagued by severe infrared divergences there was a hope that a non–zero mean of this field could serve as a natural infrared regulariser [84-87] and might help remedy the problem. The results of calculations of the effective potential in the first loops carried out in these studies found a non–zero mean of the field in the deconfinement phase. In Ref. [88] it was stated that this result is not gauge invariant and that for the the effective action, expressed in terms of the Polyakov loop, the condensation is absent, which in turn was criticised in Ref. [89]. For our presentation the ultimate resolution of this controversial issue is of no direct interest. In any case, the perturbative nature of those calculations, the lack of a consistent treatment of the surface terms and dealing with the infrared divergence in a rather careless manner — these are the defficiences which allow us to question the validity of those results. As we shall see below, the very fact that there is

a non–zero mean A_0 for a theory that correctly accounts for the surface terms, does not imply the occurrence of a spontaneous symmetry breaking, but instead corresponds to the confinement phase. We emphasise that the formal neglect of the surface terms leads to physically wrong results and mathematically incorrect computations, which is probably signalled by the difficulties encountered by the authors. The works on A_0 condensation are therefore of educational rather than practical value to us. In Ref. [84] and its follow–up papers the interpretation of A_0 condensate as the imaginary part of the chemical potential was given and it was also found that inserting the global singlet projector inside the trace does not change the value of the partition function. The mechanism of occurrence of the non–zero mean Polyakov line in the deconfinement phase in our approach is explained by the spontaneous breaking of Z_N symmetry by the surface term (which is zero in the confinement phase).

Thus, the development of the approach explaining the confinement–deconfinement phase transition in QCD from the first principles is the most important task of the theory of non–Abelian gauge fields. To justify the advantages of the present approach I may note the following circumstances. The existence of confinement means the presence of long–range interactions in the system, and this is the origin of the linearly growing potential. The reason for this phenomenon in a local theory is well understood. Indeed, due to the local gauge invariance, the system possesses the Gaussian constraint. By choosing a physical gauge, this can be resolved, expressing the auxiliary component of the electric field through the physical components of the fields in the form of an integral operator. Substituting the latter expression into the Hamiltonian of the system leads to some effectively long–ranged and nonlocal equations of motion for the physical fields.

On the other hand, the nonlocality is the source of the of the boundary nontriviality of the theory. The surface terms, containing the auxiliary component, cease to be surface terms after resolving the constraint. The boundary condition for this component acquires the status of a constraint, since it requires the nullification of a certain integral construction of the physical variables. Therefore, there is a need to develop a consistent scheme for accounting of any surface terms. The presence of non–physical degrees of freedom in a covariant formulation of the Yang–Mills theory makes the analysis of the boundary conditions problem noticeably more complicated due to the additional arbitrariness introduced by them. Thus it seems reasonable to work in some physical gauge, thereby ensuring the positive definiteness of the Hilbert space of states.

It is well known that due to the masslessness of gluons the Yang–Mills theory suffers from serious infrared divergences. In order to handle these carefully it is necessary to introduce an infrared

regularisation at the first stage, which is then removed in the final result. The simplest and most physically straightforward way to accomplish this is to regularise the system by confining it inside a finite region. Note that such a procedure is always necessary for construction of the theory at finite temperature, since, by the translational invariance of the system the logarithm of the partition function is proportional to the volume of space. Thus, we naturally arrive at the formulation of the Yang–Mills theory in a finite region with taking into account the non–trivial values of those variables at the boundary, which in the infinite limit must give rise to non–zero surface terms.

The generalised Fock–Schwinger gauge is singled out for the analysis of our problem due to its following remarkable properties:

1) The Gaussian constraint can be solved in an explicit form here;

2) By matching the gauge choice with the shape of the boundary of the domain the structure of the surface terms is significantly simplified;

3) In this gauge it is possible to relate the gauge potential fields to the strength fields by means of a linear differential relation;

4) In view of the above points, the Hamiltonian of the system turns out to be a fourth–order polynomial with respect to the canonical variables;

5) Only the longitudinal components of the chromo–electric and –magnetic fields contain non–Abelian and non–local structures. Due to this property, the functional integral can be easily rewritten in terms of only two collective variables functionally conjugated to these components;

6) The mean–field approximation is nontrivial for the collective variables due to the quasi–classical character of the background strength fields.

Despite the presence in our confinement mechanism of a background constant magnetic field at temperatures below the critical one, this circumstance does not create problems with the theory invariances, since in the zero temperature limit the mean field value vanishes and the theory becomes fully Poincaré and gauge invariant as it should. Indeed, the contribution of the surface terms leads in the confinement phase to the insertion into the mean of an invariant object — the singlet projector of the group of the large gauge transformations at infinity.

As an obvious disadvantage of the Fock–Schwinger gauge we should remark that the non–covariant character of the gauge condition and the presence of non–locality lead to a substantial complication of the renormalisation procedure. This issue will be the subject of later research and is not discussed in the thesis. We avoid the use of a formal systematic renormalisation procedure by working in terms

of renorm–invariant variables and a fixed ultraviolet cut–off. The result for the observed quantity $\xi = T_c/\sqrt{\chi}$ turns out to be finite due to the exact cancellation of divergences in it.

The outline of the thesis is as follows.

Chapter 2 introduces notations, describes the general properties and basic formulae of the generalised Fock–Schwinger gauge such as: the explicit solution of the Gaussian constraint, the linear differential relations between different variables (e.g. gauge fields and strengths), the explicit formulae for transformations between this and an arbitrary gauge, and others.

Chapter 3 is devoted to the development of the Hamiltonian formalism for the Yang–Mills theory in a finite domain, taking into account the contribution of the surface terms, detailed analysis of the equations of motion, symmetry transformations and the problem of the boundary conditions choice.

Chapter 4 presents an alternative approach to the dynamics of the Yang–Mills theory in the formalism of variables at infinity, the presentation of which is preceded by by some general information from the algebraic Quantum Field Theory (QFT). The equivalence of this formalism to the one outlined in Chapter 3 is shown, the dynamics at infinity is also derived.

Chapter 5 studies the dependence of the partition function of the Abelian and non–Abelian gauge theories on the variable at the boundary $|\chi|$. A formulation of the theory in terms of the collective variables conjugate to the longitudinal components of the strength fields is derived. The effective action of the theory in terms of the collective variables is calculated in the mean field approximation and based on its analysis a novel interpretation of the confinement–deconfinement phase transition mechanism is proposed.

In Chapter 6 it is shown that our confinement criterion satisfies the traditional criteria: the area law in the confinement phase and the spontaneous breaking of Z_N symmetry in the deconfinement phase. We calculate the ratio of the phase transition temperature to the square root of the of the string tension coefficient.

The main results of the thesis are formulated in the Conclusion.

2 Some properties of the generalised Fock–Schwinger gauge

In this Chapter we consider the procedure for fixing the generalised Fock–Schwinger (FS) gauge in the Yang–Mills theory, determine the boundary conditions on the gauge fields, which prohibit the residual transformations, obtain certain identities experessing some physical poperties in this gauge through the others, and also find the relations between the FS and Coulomb gauges.

We proceed from the necessary notations. Let $\mathbf{x}$ denote the 3-dimensional radius vector, $x = |\mathbf{x}|$ its length and $\hat{\mathbf{x}} = \mathbf{x}/x$ the unit vector in its direction. We introduce the orthogonal projector, as well as the longitudinal and perpendicular components of vectors in $\mathbf{R}^3$:

$$P = \mathbf{1} - \hat{\mathbf{x}} \otimes \hat{\mathbf{x}}, \qquad \mathbf{a}_\perp = P\mathbf{a}, \quad \mathbf{a}_\parallel = (1-P)\mathbf{a}. \tag{1}$$

The Fock–Schwinger gauge is defined by the condition

$$\hat{\mathbf{x}}\,\mathbf{A}(t,\mathbf{x}) = 0, \qquad \text{i.e. } \mathbf{A} = \mathbf{A}_\perp. \tag{2}$$

The covariant 4-dimensional version of this gauge $x_\mu A^\mu = 0$ was introduced in the classical works [90, 91], and this has found numerous applications in the non–Abelian gauge theory [92, 93]. In Ref. [94] it was noted that this gauge, along with the rather popular axial gauge, allows one to resolve the Gaussian constraint explicitly, does not contain the ghost fields, and moreover is invariant with respect to spatial rotations, which is essential for the analysis of the system eigenstates. However, the literature for some time lacked any substantial study of its main properties. Such analysis was curried out by us in papers [95, 96], and then extended to the generalised FS gauge [97], which we shall introduce now.

Let V_R be a regular domain in $\mathbf{R}^3$, topologically equivalent to a ball, with a smooth boundary ∂V_R. It is convenient to choose such a curvilinear coordinates system $\mathbf{x}$ in domain V_R that on the surface ∂V_R the first component is constant and equal to the parameter R, which will also serve as an infrared cut–off in our treatment, i.e. $\partial V_R = \{\mathbf{x} : X_1(\mathbf{x}) = R = const\}$. Recall that we denote the Cartesian coordinates by lowercase letters $\mathbf{x}$. The field of vectors normal to the boundary, taken for all values of R, forms a smooth (differentiable) vector filed in the whole domain.

The local orthonormal curvilinear basis can be presented in the following way

$$e_i^{(k)} = \frac{1}{h_k}\,\frac{\partial x_i}{\partial X_k}\,, \quad h_k = \left(\sum_{i=1}^{3}\left(\frac{\partial x_i}{\partial X_k}\right)^2\right)^{1/2}, \quad h \equiv \prod_{i=1}^{3} h_i. \tag{3}$$

In our notations we do not distinguish the upper and lower indices and summation over the repeated indices is implicitly assumed, unless otherwise stated. The components of vectors in the curvilinear basis, in order to distinguish them from those in the Cartesian basis, are denoted by letters enclosed in parentheses,

$$A_{(k)} = e^{i}_{(k)}\, A_i, \quad \partial_{(k)} \equiv \frac{1}{h_k}\,\frac{\partial}{\partial X_k}. \tag{4}$$

The vector $\mathbf{e}_{(1)}$, obviously, defines the field of normals described above. It is natural to introduce the (2+1) decomposition of vector components onto the longitudinal and transverse components (denoted by the Greek letters): $i \to (1, \alpha)$, $\alpha = 2, 3$. For the vector notation for the "Greek" curvilinear components we use the following symbols $\breve{\mathbf{X}} = (X_2, X_3)$.

The gauge field theory in a finite domain V_R acquires the simplest form in a special gauge, which is consistent with the shape of the enclosing boundary. Namely, we shall require that the gauge field component along the normals field is equal to zero in every point:

$$\mathbf{e}_{(1)}(\mathbf{x})\,\mathbf{A}(t, \mathbf{x}) = 0, \;\; \mathbf{A} = \mathbf{A}_\perp, \quad \mathbf{A}_\perp = P\mathbf{A}, \;\; P = \mathbf{1} - \mathbf{e}_{(1)} \otimes \mathbf{e}_{(1)}. \tag{5}$$

Such a gauge naturally generalises the originally proposed gauge of Ref. (2), and thus is called the generalised Fock–Schwinger gauge.

A remarkable property of this gauge is that the Gauss constraint

$$\nabla_i E_i = 0 \tag{6}$$

can be resolved in it explicitly as follows

$$E_{(1)} = -\frac{h_1}{h} \int_{X_1^{(0)}}^{X_1} dX_1'\, (h\Phi_\perp)(X_1', \breve{\mathbf{X}}), \tag{7}$$

expressing the longitudinal composition of the electric field through the transverse part of the constraint

$$\Phi_\perp \equiv \nabla_i E_{\perp\, i} = \frac{h_\alpha}{h} \nabla_{(\alpha)} \left(\frac{h}{h_\alpha} E_{(\alpha)} \right). \tag{8}$$

The lower integration limit $X_1^{(0)}$ is a constant, which, as we shall see below, is reasonable to choose at the origin of the coordinate system. In deriving (7) we have used the explicit expression for the differentiation operator $\partial \mathbf{e}_{(1)} = h^{-1} \partial_{(1)} h$ and the fact that the non–Abelian part of the constraint does not contain any longitudinal vector components due to the gauge condition.

In complete analogy, we can resolve the identity

$$\nabla_i G_i = 0, \quad G_i \equiv \nabla_j F_{ij}, \tag{9}$$

expressing $G_{(1)}$ via the transverse components of this vector,

$$G_{(1)} = -\frac{h_1}{h} \int_{\bar{X}_1^{(0)}}^{X_1} dX_1' \, (h \, \nabla_i G_{\perp i})(X_1', \breve{\mathbf{X}}). \tag{10}$$

Next, one of the components of the Bianchi identities can be written in the form

$$\nabla_i B_i = 0, \quad B_k = \frac{1}{2}\epsilon_{ijk} F_{ij}, \tag{11}$$

which analogously allows us to express the longitudinal components of the magnetic field

$$B_{(1)} = -\frac{h_1}{h} \int_{X_1^{(0)}}^{X_1} dX_1' \, (h \, \nabla_i B_{\perp i})(X_1', \breve{\mathbf{X}}). \tag{12}$$

It can be shown that the remaining two components of the Bianchi identities

$$\epsilon_{ijk} \, e^i_{(\alpha)} \nabla_j E_k = 0 \tag{13}$$

are equivalent to the following identities

$$\frac{1}{h_\alpha} \frac{\partial}{\partial X_1} \left(h_\alpha E_{(\alpha)} \right) = \nabla_{(\alpha)} \left(h_1 E_{(1)} \right), \tag{14}$$

linking the transverse electric field components with the covariant derivative of the longitudinal one.

A very attractive property of the Fock–Schwinger gauge, due to which its covariant version has often been used, is the identity relating the gauge potential field to the strength field. Such a relation also holds in the generalised FS gauge. It is a direct consequence of the gauge condition and the definition of the strength tensor [97]

$$\frac{1}{h_\alpha} \frac{\partial}{\partial X_1} \left(h_\alpha A_{(\alpha)} \right) = h_1 F_{(\alpha)(1)}. \tag{15}$$

The integral form of the identity

$$A_{(\alpha)} = \frac{1}{h_\alpha} \int_{\tilde{X}_1^{(0)}}^{X_1} dX_1' \, (h_1 h_\alpha F_{(\alpha)(1)})(X_1', \breve{\mathbf{X}}), \tag{16}$$

as it is easy to understand, breaks the symmetry with respect to the residual gauge transformations allowed by the gauge condition (5). The choice of boundary conditions in the previous integral

relations does not affect the residual symmetry, since they relate the quantities that are transformed by it uniformly.

The fixation of some boundary condition in (16) turns out to be very restrictive. Let us choose $\tilde{X}_1^{(0)} = x_1^0$ at the origin point $\mathbf{x}^0$, i.e. we impose a boundary condition of the form

$$\lim_{X_1 \to x_1^0} (h_\alpha A_{(\alpha)})(\mathbf{X}) = 0. \tag{17}$$

The following is a valid

> Statement. *For any gauge field there exists a unique element of the group of gauge transformations, which transforms it into a field satisfying the conditions (5,17), all solutions of which are connected only by homogeneous transformations, i.e. these conditions ensure the unique choice of a Lie algebra element for each gauge orbit.*

Let $\tilde{\mathbf{A}}$ be an arbitrary gauged field. The transition to the generalised FS gauge is performed by means of the gauge transformation $U(\mathbf{X})$:

$$\mathbf{A}(\mathbf{X}) = U^{-1}\,(\,\tilde{\mathbf{A}}(\mathbf{X}) - g^{-1}\,\partial\,)\,U(\mathbf{X})\,, \quad \mathbf{E}(\mathbf{X}) = U^{-1}\tilde{\mathbf{E}}(\mathbf{X})\,U(\mathbf{X}), \tag{18}$$

which can be found as the solution of the equation

$$\frac{1}{h_1}\frac{\partial}{\partial X_1} U(\mathbf{X}) = g(\mathbf{e}_{(1)}\,\tilde{\mathbf{A}})(\mathbf{X})\,U(\mathbf{X})\,. \tag{19}$$

The initial condition for now is chosen in the simplest way $U(X_1 = 0, \breve{\mathbf{X}}) = 1$. The solution of this can be written via the Dyson P-exponent

$$U(\mathbf{X}) = P\exp\int_0^1 d\alpha\, R(\alpha, \mathbf{X}), \;\; R(\alpha, \mathbf{X}) = gX_1(h_1\mathbf{e}_{(1)}\tilde{\mathbf{A}})(\alpha X_1, \breve{\mathbf{X}}). \tag{20}$$

It can be shown [96] that for the components of the gauge field the transformation formulae can be explicitly expressed as

$$\begin{aligned}\mathbf{A}^b(\mathbf{X}) = \tilde{\mathbf{A}}^a(\mathbf{X})\,P\exp\int_0^1 d\alpha\,(-gt^{abc}\,R^c(\alpha,\mathbf{X})\,) - \\ -g^{-1}\int_0^1 d\beta\,\partial R^a(\beta,\mathbf{X})\,P\exp\int_0^\beta d\gamma\,(-gt^{abc}\,R^c(\gamma,\mathbf{X})\,).\end{aligned} \tag{21}$$

By choosing the residual transformation, it is possible to satisfy the necessary boundary condition (17), which plays a key role in proving the uniqueness of the gauge field. Indeed, suppose the existence

of two distinct fields $\mathbf{A}'$ and $\mathbf{A}''$ satisfying (5,17) and such that $\mathbf{A}'' \neq U^{-1}\mathbf{A}'U$, $\forall\ U = \text{const}$. Then there must exist a gauge transformation linking them

$$\mathbf{A}''(\mathbf{X}) = U^{-1}(\mathbf{X})\ (\mathbf{A}'(\mathbf{X}) - g^{-1}\,\partial\,)\ U(\mathbf{X})\,. \tag{22}$$

Multiplication by $\mathbf{e}_{(1)}$ gives $\frac{1}{h_1}\frac{\partial\, U(\mathbf{X})}{\partial\, X_1} = 0$, i. e. $U = U(\breve{\mathbf{X}})$.

The set of transformations of this type forms the group of residual of G_{res} gauge transformations of the FS gauge. The boundary condition (17) forbids such transformations. Indeed, in the limit $\mathbf{X} \to \mathbf{x}^0$ in (22) the gauge field will have a singularity and we come to a contradiction with the boundary condition, which thus proves the uniqueness.

As a simple example, we consider the ellipsoidal coordinates $1 \leq X_1 < \infty$, $-1 \leq X_2 \leq 1$, $0 \leq X_3 \leq 2\pi$, where $X_3 = \phi$ - polar angle, $X_1 = (r_1 + r_2)/2a$, $X_2 = (r_1 - r_2)/2a$. This coordinate system is defined by two points, located at a distance $\pm a$ from the centre along the axis z, and $\mathbf{r}_1$, $\mathbf{r}_2$– radius–vectors from these points to the point of observation. The origin of the coordinates is at the point $x_1^0 = 1$, $x_2^0 = 0$. The Lamé parameters in the ellipsoidal coordinates are equal to

$$h_1^2 = a^2\frac{X_1^2 - X_2^2}{X_1^2 - 1},\ h_2^2 = a^2\frac{X_1^2 - X_2^2}{1 - X_2^2},\ h_3^2 = a^2(X_1^2 - 1)(1 - X_2^2). \tag{23}$$

In the limit $a = 0$, the ellipsoidal region becomes a ball, the coordinate system becomes spherical, and the the gauge becomes the traditional Fock–Schwinger gauge. In spherical coordinates $X_1 = r$, $X_2 = \phi$, $X_3 = \theta$ the Lamé parameters have a particularly simple form

$$h_1 = 1, \quad h_2 = X_1 \sin X_3, \quad h_3 = X_1. \tag{24}$$

An important property of the spherical FS gauge, which simplifies the formulae considerably is that $\mathbf{e}_{(i)}$ is independent of X_1. The normal vector is then equal to the unit radius–vector $\mathbf{e}_{(1)} = \hat{\mathbf{x}}$.

Finally, I shall consider the question of equivalence of the Fock–Schwinger and the Coulomb gauges in the Abelian theory in infinite volume, and also establish explicit transformation formulae between them. It is useful to introduce the longitudinal and transverse components of vectors in the momentum space, denoted by the symbols $\perp$, $\|$ as upper indices of the vectors

$$Q = \mathbf{1} - \partial \otimes \Delta^{-1}\partial\,, \qquad \mathbf{a}^{\perp} = Q\mathbf{a}\,, \quad \mathbf{a}^{\|} = (1 - Q)\mathbf{a}\,. \tag{25}$$

By the inverse Laplace operator here we mean the integral operator

$$[\,\Delta^{-1}\,\mathbf{f}\,]\ (x) = -\int d\mathbf{x}'\,(\,4\pi\mid \mathbf{x} - \mathbf{x}'\mid)^{-1}\mathbf{f}(\mathbf{x}')\,. \tag{26}$$

The Coulomb gauge

$$\partial \mathbf{A} = 0, \qquad \text{i.e. } \mathbf{A} = \mathbf{A}^{\perp} \tag{27}$$

is naturally singled out in the Abelian theory in the infinite volume, since its variables diagonalise the free Hamiltonian. We need to introduce the following integral operators in $\mathbf{R}^3$

$$[\,\mathrm{K}\,\mathbf{f}\,]\,(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - \,\partial \int_0^x dy\, \hat{\mathbf{x}}\mathbf{f}(y\hat{\mathbf{x}})\,, \tag{28}$$

$$[\,\mathrm{G}\,\mathbf{f}\,]\,(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - \hat{\mathbf{x}}\, x^{-2} \int_0^x y^2 dy\,[\,\partial\,\mathbf{f}\,](y\hat{\mathbf{x}}), \tag{29}$$

satisfying the simple algebra [96]

$$\mathrm{KP} = \mathrm{P}\,, \quad \mathrm{PK} = \mathrm{K}\,, \quad \mathrm{QK} = \mathrm{Q}\,, \quad \mathrm{KQ} = \mathrm{K}\,, \tag{30}$$

$$\mathrm{GP} = \mathrm{G}\,, \quad \mathrm{PG} = \mathrm{P}\,, \quad \mathrm{QG} = \mathrm{G}\,, \quad \mathrm{GQ} = \mathrm{Q}\,. \tag{31}$$

Let $\mathcal{X}_{\|}$, $\mathcal{X}_{\perp}$ and $\mathcal{X}^{\|}$, $\mathcal{X}^{\perp}$ denote the longitudinal and transverse subspaces of $\mathbf{R}^3$ in the sense of transversality in the coordinate and the momentum spaces. Consider the Gauss law contraint in the presence of an external charge density

$$\partial \mathbf{E} = \rho. \tag{32}$$

The total strength is gauge–invariant in the Abelian case and is expressed through the corresponding transverse components in both gauges

$$\mathbf{E} = \mathbf{E}_{\perp} - \frac{\hat{\mathbf{x}}}{x^2} \int_0^x y^2 dy\,(\partial \mathbf{E}_{\perp} - \rho)(y\hat{\mathbf{x}}) = \mathbf{E}^{\perp} + \partial \Delta^{-1} \rho. \tag{33}$$

Under the assumption of boundary conditions of the form:

$$\lim_{x\to 0} x\,\mathbf{A}_{\perp}(\mathbf{x}) = 0, \quad \lim_{x\to 0} x^2 E^{\perp}(\mathbf{x}) = 0 \tag{34}$$

we can prove that the mappings defined by the operators

$$\begin{aligned} K:&\quad \mathcal{X}^{\perp} \to \mathcal{X}_{\perp}, & \qquad Q:&\quad \mathcal{X}_{\perp} \to \mathcal{X}^{\perp} \\ P:&\quad \mathcal{X}^{\perp} \to \mathcal{X}_{\perp}, & \qquad G:&\quad \mathcal{X}_{\perp} \to \mathcal{X}^{\perp} \end{aligned}$$

are one-to-one transformations between the variables of the two gauges

$$\begin{aligned} \mathbf{A}_{\perp} &= K\mathbf{A}^{\perp}, & \qquad \mathbf{A}^{\perp} &= Q\mathbf{A}_{\perp}, \\ \mathbf{E}_{\perp} &= P(\mathbf{E}^{\perp} + \partial\Delta^{-1}\rho), & \qquad \mathbf{E}^{\perp} &= G(\mathbf{E}_{\perp} - \partial\Delta^{-1}\rho). \end{aligned} \tag{35}$$

These linking formulae for the gauge field follow directly from the relations (18,19) (or (33)) and similar relations for the transition to the Coulomb gauge, and for the electric field from (33) formula. The functional Jacobian of the transition from one set of variables to another is independent of the variables themselves and equal to some normalisation constant $\mathcal{D}A_\perp\,\mathcal{D}E_\perp = N\,\mathcal{D}A^\perp\,\mathcal{D}E^\perp$ [96]. Using the algebra of operators (30,31) we can see that this transformation is canonical, i.e. it preserves the Poisson brackets

$$\{\, A^i_\perp(\mathbf{x}), E^j_\perp(\mathbf{y})\,\} = (\,\mathrm{KQP}\,)_{ij}(\mathbf{x},\mathbf{y}) = \mathrm{P}_{ij}\,\delta(\mathbf{x}-\mathbf{y}). \tag{36}$$

In Appendix A we present the formula for the propagator in the FS gauge, which is derived using the connection formulae we have used for the Coulomb gauge variables. We emphasise that the latter ones are in agreement with the FS gauge variables in the sector restricted by a fixed boundary condition of the form (34). By choosing a different lower limit in the formula (28) one can establish the kind of transformation to an arbitrary sector of the FS gauge. In other words, the formulation of the theory in the FS gauge has additional degrees of freedom numbering different topological sectors, which are transformed into each other by the action of the elements of the residual symmetry group. This fact does not lead to any physical consequences in the Abelian theory, where the integral over the orbits of the group gives only a normalisation factor in the functional integral, but may turn out to be essential in the non–Abelian theory.

3 Hamiltonian formalism of the Yang–Mills theory in a finite domain

The standard assumption of the Hamiltonian formalism in the field theory is the absence of surface terms arising from integration by parts. For problems in a finite domain this is ensured by a proper choice of the boundary conditions satisfied by the fields at the surface. In the limit of infinite volume it is sufficient to assume that the fields decrease sufficiently fast at the spatial infinity. Such requirements are certainly well justified for many problems in the field theory. Theory with a gauge symmetry, however, has a number of specific properties. First of all, in its covariant formulation it contains superfluous (unphysical) degrees of freedom. The standard assumptions about the behaviour of these components at infinity is, strictly speaking, not obvious. Indeed, if one chooses a physical gauge, the extra component of the electric field after the resolution of the Gaussian constraint is expressed in

the form of a volume integral of the physical variables. In other words, this *turns the surface terms into volume expressions*, and therefore there is no reason to neglect them. The boundary condition on such a variable, therefore, rather has the meaning of an additional constraint (the constraint at the surface in the original formulation). But as we know, constraints do require a more careful analysis. Since, as a rule, we are not able to explicitly resolve such a relation, the equations of motion for the physical variables must be derived without its use, and only then we should impose it on the equations of motion. A simple way to account for a surface constraint is to add it to the Hamiltonian with a new Lagrange multiplier.

Thus, it is necessary to derive the equations of motion while taking into account the surface terms, but in addition, the values of the fields at the boundary should appear as Hamiltonian variables on equal footing. Moreover, the addition of the surface term to the Hamiltonian, necessary for taking into account the surface constraint, acquires a significant physical meaning. These features of the theory appear rather unusual. Nevertheless, such a formalism can be quite easily constructed for the Yang–Mills theory, which is the main purpose of this Chapter.

On the other hand, we can avoid these problems completely if we work with the Hamiltonian after excluding the non–physical variables. But in this case, the functional becomes effectively a nonlocal expression and the theory contains nonlocal long–range interactions. The Hamiltonian formalism can be made well–defined in the traditional sense by introducing into the Hamiltonian a smooth cut–off function with a bounded support [98, 99, 100]. Due to this function all surface terms turn to zero, but at a price — the equations of motion contain a dependence on the cut–off function. In the limit, when the support of the function becomes infinite, the equations of motion of the local variables still contain ill–defined delocalised variables. Thus, the problem of accounting for infinity persists, although it takes a slightly different form.

The second approach described above is quite feasible for the Yang–Mills theory [95, 96] and is discussed in the next Chapter, but the first one has clear advantages in its simplicity and elegance. Moreover, the system regularised by a smooth cut–off does not have a direct physical meaning, while the problem about a system inside a finite domain is quite natural. In the infinite limit both approaches, of course, become equivalent, but the procedure of taking the limit presents certain difficulties by itself.

The problem of accounting for the boundary in the Hamiltonian formalism naturally arises, for example, when considering surface waves in hydrodynamics [101, 102, 103], where the surface terms

have a direct physical meaning. In the work of Regge and Teitelboim [104] it was shown that the surface terms can play an essential role in the theory of gravitation, where this question is still a subject of active study [105, 106].

For a certain class of field models, in Ref. [107] a generalisation of the Hamiltonian formalism was proposed, the main idea of which is to modify the Poisson brackets by the addition of some surface terms so that they are strictly (and not with accuracy up to the surface terms, as in the traditional formalism), satisfy the Jacobi identities.

We restrict ourselves to the formulation for the case when the Hamiltonian of the system contains derivatives of canonical variables of at most first order. Consider the class of local functionals of the canonical variables

$$F = \int_V d\mathbf{x}\ f[\varphi^A;\, \varphi^A_{,i}]. \tag{37}$$

Let the canonical variables possess the canonical Poisson bracket

$$\{\varphi^A(\mathbf{x}), \varphi^B(\mathbf{x}')\} = I^{AB}\,\delta(\mathbf{x}, \mathbf{x}'), \tag{38}$$

where I^{AB} is the standard samplectic matrix with the property: $I^2 = -1,\ I = const.$ Variation of this functional with respect to a change of the canonical variables can be written as

$$\delta F = \int_V d\mathbf{x}\ \Big(\mathcal{E}_A(F)\,\delta\varphi^A + \partial_i\big(\mathcal{E}^i_A(F)\,\delta\varphi^A\big) \Big), \tag{39}$$

where the Euler derivatives of the zeroth and first orders are equal to respectively

$$\mathcal{E}_A(F) = \frac{\partial f}{\partial \varphi^A} - \partial_i \frac{\partial f}{\partial \varphi^A_{,i}}\,, \quad \mathcal{E}^i_A(F) = \frac{\partial f}{\partial \varphi^A_{,i}}. \tag{40}$$

We define the Poisson bracket of two arbitrary functionals F and H as:

$$\begin{aligned} \{F, H\} \;=\; & \int_V d\mathbf{x}\Big(\mathcal{E}_A(F)\, I^{AB}\, \mathcal{E}_B(H) \\ & + \partial_i\big(\mathcal{E}^i_A(F)\, I^{AB}\, \mathcal{E}_B(H) + \mathcal{E}_A(F)\, I^{AB}\, \mathcal{E}^i_B(H)\big) \\ & + \partial_i\partial_j\big(\mathcal{E}^i_A(F)\, I^{AB}\, \mathcal{E}^j_B(H)\big)\Big). \end{aligned} \tag{41}$$

If H is the Hamiltonian of the system, then the time evolution of functional F is governed by the differential equation

$$\dot{F} = \{F, H\}. \tag{42}$$

This Poisson bracket has the required properties of antisymmetry, completeness and satisfies the Jacobi identities of [107]. We can see that the equations of motion of the form (41,42) clearly follow from the variational principle of action with full accounting for the surface terms

$$\begin{aligned} S &= \int_V d\mathbf{x} \left(\frac{1}{2} \dot{\varphi}^A \, I^{-1}_{AB} \, \varphi^B - h(\varphi^A; \varphi^A_{,i}) \right), \qquad (43) \\ \delta S &= \int_V d\mathbf{x} \left(\dot{\varphi}^A \, I^{-1}_{AB} \, \delta\varphi^B - \mathcal{E}_B(H) \delta\varphi^B \right. \\ &- \left. \partial_i \big(\mathcal{E}^i_B(H) \delta\varphi^B \big) \right). \qquad (44) \end{aligned}$$

Let us apply the described formalism to the Yang–Mills theory. We consider the Hamiltonian of the system in a finite region V

$$\begin{aligned} \mathcal{H}_V &= \int_V d\mathbf{x} \left(\frac{1}{2} (E^a_i)^2 + \frac{1}{4} (F^a_{ij})^2 \right. \\ &- \left. A^a_0 \left(\partial_i E^a_i - g t^{abc} A^b_i E^c_i \right) \right). \qquad (45) \end{aligned}$$

When carrying out the reduction of the system to physical variables in the generalised Fock–Schwinger gauge we have seen that the variable $E_{(1)}$ is not independent. Therefore any boundary condition on it is an additional constraint. We modify the Hamiltonian $H_V = \mathcal{H}_V + \Delta H_V$ by the addition of a surface term containing this constraint with the surface Lagrange multiplier $A_0(R\hat{\mathbf{x}})$,

$$\Delta H_V = \int_{\partial V} d\breve{\mathbf{X}} \; A^a_0 \left(\frac{h}{h_1} E^a_{(1)} + \chi^a \right), \qquad (46)$$

where $\chi = \chi(\breve{\mathbf{X}})$ is an arbitrary function specifying the boundary condition, which we do not specify yet. Note that without the addition of this surface term, the equations of motion in a finite domain are not localisable.

Assuming the usual canonical Poisson bracket $I_{A^a_i E^b_j} = \delta^{ab}\delta_{ij}$, it is easy to obtain the following equations of motion

$$\int_V d\mathbf{x} \, g^a_i \dot{A}^a_i = \int_V d\mathbf{x} \, g^a_i (E^a_i + \nabla_i A^a_0), \qquad (47)$$

$$\begin{aligned} &\int_V d\mathbf{x} \, g^a_i \dot{E}^a_i = \int_V d\mathbf{x} \, g^a_i (-g \, t^{abc} E^b_i A^c_0 + \nabla_j F^a_{ij}) \\ &- \int_{\partial V} d\breve{\mathbf{X}} \, g^a_{(\alpha)} \; \frac{h}{h_1} \left(\frac{1}{h_\alpha} \partial_{(1)} (h_\alpha A^a_{(\alpha)}) - \frac{1}{h_1} \partial_{(\alpha)} (h_1 A^a_{(1)}) \right). \qquad (48) \end{aligned}$$

In deriving the surface term we have used the following identities for the derivatives of the basis vectors

$$e_j^{(k)}\partial_{(k)}e_j^{(\alpha)} = (1-\delta^{k\alpha})\frac{1}{h_k h_\alpha}\frac{\partial h_k}{\partial X_\alpha}, \tag{49}$$

$$e_j^{(\beta)}(\partial_{(\alpha)}e_j^{(1)} - \partial_{(1)}e_j^{(\alpha)}) = \delta^{\alpha\beta}\frac{1}{h_1 h_\alpha}\frac{\partial h_\alpha}{\partial X_1}. \tag{50}$$

Integrals of the test functions cannot be removed here, i.e. the time evolution is defined only for distributions. The equation of motion for the variable canonically conjugate to A_0 gives the Gaussian constraint

$$\begin{aligned} 0 &= \int_V d\mathbf{x}\, g^a\, \nabla_i E_i^a \\ &- \int_{\partial V} d\breve{\mathbf{X}}\, g^a \left(\frac{h}{h_1}E_{(1)}^a + \chi^a\right). \end{aligned} \tag{51}$$

Due to the independence of the values of test functions inside and on the boundary this splits into the two independent conditions

$$\nabla_i E_i^a = 0, \tag{52}$$

$$\frac{h}{h_1}E_{(1)}^a(R,\breve{\mathbf{X}}) = -\chi^a(\breve{\mathbf{X}}). \tag{53}$$

Now we are going to fix the generalised Fock–Schwinger gauge (5).

Note that Eq. (47) is essentially local, so we can treat it as an equation for a function rather than that for a distribution. We can apply to the both sides of the transverse part the operation $\frac{1}{h_1 h_\alpha}\frac{\partial}{\partial X_1}h_\alpha$. By means of the identities (50) and a relationship, which is the longitudinal part of (47),

$$\frac{1}{h_1}\frac{\partial A_0^a}{\partial X_1} = -E_{(1)}^a \tag{54}$$

we obtain the following equation of motion

$$\dot{F}_{(\alpha)(1)}^a = -gt^{abc}F_{(\alpha)(1)}^b A_0^c, \tag{55}$$

where for brevity we have utilised identity (15). Similarly, the equation of motion is local for the longitudinal component in (48) and we get

$$\dot{E}_{(1)}^a = -gt^{abc}E_{(1)}^b A_0^c + G_{(1)}^a. \tag{56}$$

By means of direct, though somewhat cumbersome calculations, by using Eqs. (49,50), one can prove the following identity for the longitudinal component of vector $G_i = \nabla_j F_{ij}$,

$$G_{(1)} = -\frac{h_1 h_\alpha}{h} \nabla_{(\alpha)} \left(\frac{h}{h_1 h_\alpha} F_{(\alpha)(1)} \right). \tag{57}$$

We are now ready to consider the problem of choosing the boundary conditions. In Eq. (12), following from Bianchi's identities, as well as in the relation expressing the longitudinal component of the electric field via the local Gaussian part (7), we require the usual conditions of regularity of the fields at the origin $\mathbf{x}^0$ (i.e. we choose the constant $X_1^{(0)} = x_1^0$):

$$\frac{h}{h_1} B_{(1)}(\mathbf{X}) \Big|_{X_1 = x_1^0} = 0, \tag{58}$$

$$\frac{h}{h_1} E_{(1)}(\mathbf{X}) \Big|_{X_1 = x_1^0} = 0, \tag{59}$$

In relations (10) for $G_{(1)}$, and in the non–local part of the constraint (53) we choose the constant $\tilde{X}_1^{(0)} = R$, which gives the boundary conditions at the surface:

$$\frac{h}{h_1} G_{(1)}(\mathbf{X}) \Big|_{X_1 = R} = 0, \tag{60}$$

$$\frac{h}{h_1} E_{(1)}(\mathbf{X}) \Big|_{X_1 = R} = -\chi(\breve{\mathbf{X}}), \tag{61}$$

i.e. thanks to Eq. (57),

$$F_{(\alpha)(1)}(R, \breve{\mathbf{X}}) = 0. \tag{62}$$

With this boundary condition on $F_{(\alpha)(1)}$, the equations of motion for the physical variables (47,48) become local. Lagrange multiplier A_0 is determined from Eq. (54)

$$A_0(\mathbf{X}) = A_0(R, \breve{\mathbf{X}}) - \int_R^{X_1} dX_1' \, (h_1 \, E_{(1)})(X_1', \breve{\mathbf{X}}) \tag{63}$$

only up to the surface Lagrange multiplier, which, in general, remains arbitrary.

This arbitrariness reflects the symmetry of the original action with respect to the time–dependent gauge transformations and is analogous to the arbitrariness in the choice of A_0 in the Coulomb gauge. It is fixed by setting the boundary conditions with respect to time, such as the periodic ones in the partition function at finite temperature. Therefore, in the temperature theory, in general, one cannot turn the surface Lagrange multiplier to zero by the choice of the gauge transformations, i. e . such a

theory contains a natural order parameter $A_0(\breve{\mathbf{X}})$. In our consideration, it will be assumed only that this quantity is a constant in time.

From equations (55,56) we see that all boundary conditions, other than surface constraint, are time–preserving and, thus, are admissible. However, from the formula (56) we obtain the law of variation of $\chi(\breve{\mathbf{X}})$ in time

$$\dot{\chi}^a(\breve{\mathbf{X}}) = gt^{abc}\, A_0^b(R, \breve{\mathbf{X}})\, \chi^c(\breve{\mathbf{X}}). \tag{64}$$

In the particular choice of the remaining arbitrariness, when both the colour vectors $\chi(\breve{\mathbf{X}})$ and $A_0(R, \breve{\mathbf{X}})$ have the same (opposite) direction at the initial time instant, χ is constant. In the general case, the boundary condition cannot be fixed and varies according to the formula (64).

The contradiction is easily eliminated by the expansion of the phase space of the system. Suppose that χ are additional Hamiltonian variables. Their Poisson brackets with ordinary variables are equal to zero, and among themselves are defined by the canonical Poisson bracket [108]

$$\{\chi^a(\breve{\mathbf{X}}), \chi^b(\breve{\mathbf{X}}')\} = \omega^{ab}[\chi]\, \delta(\breve{\mathbf{X}}, \breve{\mathbf{X}}') \equiv gt^{abc}\, \chi^c(\breve{\mathbf{X}})\, \delta(\breve{\mathbf{X}}, \breve{\mathbf{X}}'). \tag{65}$$

Then for arbitrary functionals F, G the Poisson bracket is

$$\{F, G\} = \int_{\partial V} d\breve{\mathbf{X}}\, \omega^{ab}[\chi] \frac{\delta F}{\delta \chi(\breve{\mathbf{X}})}\, \frac{\delta G}{\delta \chi(\breve{\mathbf{X}})} \tag{66}$$

The Jacobi identity for such brackets follows from the Jacobi identity in Lie algebra. The phase space Γ_χ of these variables, according to the symplectic construction of Berezin–Kirillov [108], are the orbits of the co–adjoint representations of the gauge group $SU(N)$. On the example of the group $SU(2)$ we note that the symplectic spaces Γ_χ are spheres of radius $|\chi|$ in the co–adjoint the representation of the group. The radius of the sphere is the integral of the motion and represents an external parameter of the theory, as well as the frequency of the of rotations of the dynamics on the boundary $|A_0(R, \breve{\mathbf{X}})|$.

The symplectic matrix ω^{ab} is non–degenerate on the orbits, and the differentiable form $\prod_{\breve{\mathbf{X}}} (\omega^{-1})_{ab} d\chi^a(\breve{\mathbf{X}}) \bigwedge d\chi^b(\breve{\mathbf{X}})$ is a symplectic matrix invariant with respect to the Hamiltonian phase flow (Liouville's theorem). Then the equation (64) naturally coincides with the Hamilton equation for the given variables, and it is expressed through the action of the group in the co–adjoint representation. The total phase space of the system is a direct product $\tilde{\Gamma} = \Gamma \times \Gamma_\chi$, where Γ is the phase the space of localised variables.

The system has residual invariance with respect to the group G_{res} of residual gauge transformations. The generators of these transformations are the surface integrals

$$Q_V(g) = \int_{\partial V} d\breve{\mathbf{X}}\, g^a(\breve{\mathbf{X}}) \left((\frac{h}{h_1} E^a_{(1)})(R, \breve{\mathbf{X}}) + \chi(\breve{\mathbf{X}}) \right). \tag{67}$$

Indeed, they possess the algebra

$$\{Q_V(g), Q_V(g')\} = Q_V(-[g, g']), \tag{68}$$

where $[g, g']^a = g t^{abc} g^b g^c$ denotes the colour commutator. The Poisson bracket of the generators with the Hamiltonian, as we have just proved, is zero

$$\{Q_V(g), H_V\} = 0. \tag{69}$$

The action of the generators on the gauge field has the form

$$\{Q_V(g), A_{(\alpha)}(\mathbf{X})\} = \nabla_{(\alpha)} g(\breve{\mathbf{X}}). \tag{70}$$

For every fixed $A_0(R, \breve{\mathbf{X}})$, we have a certain representation of the dynamics. This representation, however, does not have a fixed value of $\chi(\breve{\mathbf{X}})$, which evolves in time. The action of the symmetry group G_{res} transforms one dynamics representation into another by changing the direction of the colour vector $A_0(R, \breve{\mathbf{X}})$. Thus, a minimal representation of the Hamiltonian, which is invariant with respect to symmetry transformations and time evolution, turns out to be a direct integral of all possible representations with different directions of A_0. The values of the vector lengths χ and A_0 are thus are fixed in each such representation and are natural order parameters numbering the irreducible representations of the dynamics.

To conclude this Chapter, I would like to consider the expression of the Hamiltonian systems in terms of the physical variables

$$\begin{aligned} H_R &= \frac{1}{2} \int_V d\mathbf{X}\, h\, (E^2_{(\alpha)} + B^2_{(k)}) \\ &+ \int_V d\mathbf{X}\, h\, \Phi_\perp \sigma + \int_{\partial V} d\breve{\mathbf{X}}\, A_0(R, \breve{\mathbf{X}}) \chi(\breve{\mathbf{X}}), \end{aligned} \tag{71}$$

$$\sigma(\mathbf{X}) = -A_0(R, \breve{\mathbf{X}}) + \int_R^{X_1} dX'_1\, (h_1 E_{(1)})(X'_1, \breve{\mathbf{X}}), \tag{72}$$

where $E_{(1)}$ is given by formula in Eq. (7).

The variable at the boundary $\sigma_R(\breve{\mathbf{X}}) = -A_0(R, \breve{\mathbf{X}})$ plays an important role in the following considerations. Introducing this notation is useful because it allows us to rewrite the bulk part of the the Hamiltonian as a local expression by introducing the collective variable σ.

4 Yang–Mills theory in the variables at infinity formalism

In the previous Chapters we have considered the construction of a a closed Hamiltonian formalism for the Yang–Mills theory in a finite domain, allowing for non–trivial values of the variables at the boundary. The phase space of the system has been extended by introducing additional degrees of freedom at the boundary. In doing so, the Poisson bracket had to be modified by adding some surface terms so that it still satisfied the Jacobi identities. As has been shown, the Poisson bracket thus introduced follows from the variational principle of action in a finite region, and is therefore uniquely defined.

The choice of the surface part of the Hamiltonian was dictated by the requirement of localisation of the dynamics, and this requirement was satisfied by the presence of additional surface constraints. It is natural to expect that in the infinite limit the dynamics in the finite region tended to be the same dynamics which arises in the limit of the smooth cut–off formulation of the system. The latter regularisation is mathematically well–defined, is unique and contains no additional surface degrees of freedom. The procedure for taking its limit turns out to be essentially nontrivial for systems with long–range interactions and must be realised in a weak topology, i.e., with reference to some narrow class of admissible physical states.

The mathematically rigorous basis for the present consideration is given by the formalism of variables at infinity, proposed by Morcchio and Strocchi in the framework of the algebraic QFT [98, 99, 100], the formulation of which we shall preface with some definitions and the results of a mathematical nature [109, 110] necessary for understanding the substance of the subject.

4.1 Some background from the formalism of the variables at infinity

In the context of algebraic QFT (Quantum Field Theory), observables correspond to self–adjoint elements of the C^* algebra $\mathcal{R}$ (a complex Banach algebra with involution and the property $|| R ||^2 = || R^* R ||$ $\forall\ R \in \mathcal{R}$), and the states — positive ($\phi(A^* A) \geq 0$) linear functionals ϕ over it, i.e., the elements of the dual of the space $\mathcal{R}^*$. In QFT, a quasi–local algebra is usually used $\mathcal{A} = \overline{\bigcup_{V \in \Im} \mathcal{A}^V}$ defined as C^*-inductive limit of algebras $\mathcal{A}^V$ associated to finite regions from the set $\Im = \{V : V \in \mathbf{R}^3\}$. The dynamics is the one-parameter group α_V^t of automorphisms of $\mathcal{A}$, generated by the truncated Hamiltonian $H_V \in \mathcal{A}$:

$$\alpha_V^t[A] \equiv e^{i\,t H_V}\, A\, e^{-i\,t H_V}\,, \quad \forall\ A \in \mathcal{A}\,. \tag{73}$$

Any C^* algebra can be considered as a W^* algebra (von Neumann algebra $\mathcal{N}$, which is a subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators in some Hilbert space $\mathcal{H}$ and such that $\mathcal{N}'' = \mathcal{N}$. Here the dash corresponds to taking the commutant $\mathcal{N}$, i.e., the set of all operators in $\mathcal{B}(\mathcal{H})$, commuting with all elements of $\mathcal{N}$.

There is a canonical GNS (Gelfand–Naimark–Segal) construction to produce a vector representation of π_φ of the algebra $\mathcal{R}$, corresponding to any state $\varphi \in \mathcal{R}^*$ in a certain Hilbert space $\mathcal{H}_\varphi$. Thus the vector $\Phi \in \mathcal{H}_\varphi$ associated with φ and defined as $\langle \varphi, R\rangle = (\Phi, \pi_\varphi(R)\Phi) \quad \forall\, R \in \mathcal{A}$ turns out to be cyclic in this representation. Universal representation $\pi_u = \bigoplus_{\varphi\in\mathcal{R}^*} \pi_\varphi$ is the direct sum of GNS representations in the Hilbert space $\mathcal{H}_u = \bigoplus_{\varphi\in\mathcal{R}^*} \mathcal{H}_\varphi$ over all states over the algebra.

The universal von Neumann algebra $\mathcal{R}'' \equiv \pi_u(\mathcal{R})''$, endowed with the strong topology, is a Banach dual space to the set $\mathcal{F}$, called the pre–dual, and forming a closed subspace of $(\mathcal{R}'')^*$. The elements of $\mathcal{F}$, which are called the normal states over $\mathcal{R}''$, can otherwise be further characterised as density matrices, i.e.

$$\exists\, \rho \in \mathcal{B}(\mathcal{H})\,, \quad \mathrm{Tr}_{\mathcal{H}}(\rho) < \infty\,, \quad \langle \varphi, A\rangle = \mathrm{Tr}_{\mathcal{H}}(\rho\, A) \;\; \forall\, A \in \mathcal{R}''\,.$$

Two representations π_1 and π_2 of C^* of the algebra $\mathcal{R}$ are quasi–equivalent (we write this as $\pi_1 \approx \pi_2$), if any state $\omega \in \mathcal{R}^*$ representable as normal on $\pi_1(\mathcal{R})'$:

$$\exists\, \rho \in \pi_1(\mathcal{R})'' \;\; \text{so that} \quad \langle \omega, A\rangle = \mathrm{Tr}_{\mathcal{H}_1}(\rho\, A) \;\; \forall\, A \in \mathcal{R}\,,$$

is just as normally representable on $\pi_2(\mathcal{R})'$ and vice versa. Simply put, representations of $\pi_1 \approx \pi_2$, if they have the same set of of algebra states admitting a density matrix on their bicommutants.

A primary (factor) representation is a representation, quasi–equivalent to irreducible. Primary representations will play an important role due to the following property. Centre $\mathcal{Z}_\pi = \pi(\mathcal{R})' \bigcap \pi(\mathcal{R})$ of any primary representation π is trivial and is of the form $\mathcal{Z}_\pi = \{\lambda\, \mathbf{1}, \quad \lambda \in \mathbf{C}\}$. The same property holds for irreducible representations also, but they, unlike the primary ones, are obliged to be associated with pure states.

We shall consider systems with long–range interactions for which the limit of the dynamics (73) at $V \to \infty$ does not exist in the topology of the norms. In this case the dynamics in the infinite volume can be formulated using the construction of Morchio and Strocchi [28-31].

Let $\mathcal{F}$ be some class of states from $\mathcal{A}^*$ satisfying the the properties:

i) $\mathcal{F}$ is closed with respect to linear operations;

ii) $\mathcal{F}$ is norm–closed and separable: from $\phi(A) = 0 \;\; \forall\, \phi \in \mathcal{F} \;\; \Rightarrow A = 0$;

iii) From $\phi \in \mathcal{F} \;\Rightarrow\; \phi_{AB}(\,.\,) \equiv \phi(A\,.\,B) \in \mathcal{F} \quad \forall A, B \in \mathcal{A}$.

We denote by $w_{\mathcal{F}}$ the weak topology defined by $\mathcal{F}$ on $\mathcal{A}''$, and let $\mathcal{M}$ be an extension of $\mathcal{A}$ in $w_{\mathcal{F}}$-topology[1]

$$\mathcal{A} \subset \mathcal{M} \subset \mathcal{A}''\,.$$

In abstract terms, the W^*-algebra $\mathcal{M}$ is dual to the Banach space $\mathcal{F}$. Then there exist weak limits

$$w_{\mathcal{F}}\text{-}\lim_{V\to\infty} \alpha^t_V[A] \equiv \alpha^t[A] \in \mathcal{M} \quad \forall\; A \in \mathcal{A} \tag{74}$$

and α^t can be extended to a single parametric automorphisms group $\mathcal{M}$ [99, 100].

The dynamical system is the triple $(\,\mathcal{M},\; \mathcal{F},\; \alpha^t\,)$. It is important to note that in the presence of long–range interactions the algebraic dynamics can only be defined on an algebra $\mathcal{M}$ with a nontrivial centre $\mathcal{Z} = \mathcal{M} \bigcap (\cap_V \mathcal{A}')$, generated by the variables at infinity.

We say that the factor representation π associated with a state of $\mathcal{F}$ and stable with respect to α^t, leads to an effective localisation of the of dynamics if there exists an effective localisation subalgebra $\mathcal{A}_l \subset \mathcal{M}$ with the property

i) $\mathcal{A}_l$ is exactly representable in π;

ii) $\mathcal{A}_l$ is weakly dense in $\mathcal{M}$;

iii) there exist automorphisms $\alpha^t_\pi \quad \mathcal{A}_l$ such that

$$\forall \phi \in \pi: \quad \phi(\alpha^t[A]) = \phi(\alpha^t_\pi[A]) \quad \forall\, A \in \mathcal{A}_l.$$

Hence the centre of $\mathcal{A}_l$ is trivial and there are no delocalised variables in it.

The symmetry of the system $(\,\mathcal{M},\; \mathcal{F},\; \alpha^t\,)$ is naturally defined as the group β^ς of automorphisms of $\mathcal{M}$ obtained by the of the centre of $\mathcal{A}_l$ and commuting with the dynamics[2] $\beta^\varsigma\, \alpha^t = \alpha^t\, \beta^\varsigma$. For physical reasons, it would seem sufficient to restrict ourselves to the representation of the effective localisation and the dynamics of α^t_π. However, the presence of the β^ς symmetry introduces additional complexity, because it does not commute with α^t_π, although it does commute with algebraic dynamics.

[1] $\mathcal{A}''$ as a Banach space is isomorphic to the doubly dual $\mathcal{A}^{**}$.

[2] For this it is sufficient to be commuting with the dynamics at finite cut–off.

In typical examples, it turns out that α^t_π does not leave invariant representations of the form $\pi_{(\beta^\varsigma)^*\phi}$. Therefore, we have to consider a set of states $\mathcal{S} = \{\, \phi^\varsigma = (\beta^\varsigma)^*\phi \,\}$, which is already stable under the by the action $(\alpha^t_\pi)^*$, and to introduce a representation $\Pi_\mathcal{S}$ which is a direct sum of all representations $\Pi_\mathcal{S}$ by the direct sum of all representations $\mathcal{A}_l$ obtained by the GNS construction based on the states ϕ_ς.

The representation $\Pi_\mathcal{S}$ is reducible and has a nontrivial centre $\mathcal{Z}_\mathcal{S}$ generated by variables at infinity. Dynamics α^t_π uniquely extends up to an action on $\mathcal{Z}_\mathcal{S}$, whose nontrivial form leads to some interesting physical effects. Note also that it is the dynamics of α^t_π that arises in the context of considering the of the given representation of $\Pi_\mathcal{S}$.

Allow me to explain the necessity of introducing such mathematical structures. If the interaction is not decreasing fast enough at infinity, there is no limit (by the norm) of the dynamics as the finite cut is removed. This leads to the fact that the time derivative of the dynamics, i.e., the equations of motion of the local observables, symbolically written in the form

$$\frac{d}{dt}\,\alpha^t_R[\varphi(x)] = F[\varphi(x), \varphi_R], \quad \varphi(x) \in \mathcal{A} \tag{75}$$

contain delocalised variables φ_R in the limit $R \to \infty$. For example:

1° $\varphi_R = -\int_0^\infty dx\, f'_R(x)\,\varphi(x)$, where $f_R(x)$ is a smooth function such that

$$f_R(x) = \begin{cases} 1, & x \le R' \qquad R' = (1-\varepsilon)R \\ 0, & x \ge R'' \qquad R'' = (1+\varepsilon)R\,. \end{cases} \tag{76}$$

Such variables arise in field–theoretic models.

2° $\varphi_V = \frac{1}{V}\int_V dx\,\varphi(x)$ — ergodic averages. This example is typical of models in the mean–field approximation. fields, where such constructions can be present directly in the the Hamiltonian H_V.

The limits of these variables are obviously well defined in the weak sense. Let ϕ_λ is a state such that $\langle \phi, \varphi(x)\rangle = \lambda$. Then

$$w\text{-}\lim_R \varphi_R \equiv \lim_R \langle \phi^\lambda, \varphi_R\rangle = -\int_0^\infty dx\, f'_R(x)\lambda = \lambda,$$

and analogously for φ_V. Therefore, the weak limit of the dynamics $w\text{-}\lim_R \alpha^t_R \equiv \alpha^t$ exists.

The dynamics of the α^t_π of the representation with an effective localisation of the variable at infinity, corresponding to some observable, contains quite important information about the spectral

properties of the theory as this is related, by the generalised Goldstone's theorem [98], to the spectrum of the commutator of the symmetry generator with the observable in question. To find the dynamics at infinity, in general, it is necessary to solve the full dynamical problem. Let $\{\varphi^i(x)\}$ be the set of local variables of the theory, and $\{\varphi^A\}$ variables from this set, for which in the equations of motion

$$\frac{d}{dt}\alpha_R^t[\varphi^i(x)] = F^i[\varphi^j(x), \varphi_R^A] \tag{77}$$

contain delocalised contributions.

It is necessary to introduce a set of variables at infinity φ_∞^A and a set of primary states $\{\phi_\lambda\}$ such that $(\alpha_\pi^t)^*\phi_\lambda = \phi_{\lambda(t)}$ is contained in this set, each of which possesses the property

$$\langle \phi_\lambda,\, \varphi^A(x)\rangle = \lambda^A\,. \tag{78}$$

By definition, in order to obtain $\alpha_\pi^t[\varphi_\infty]$ it is necessary, by solving Eq. (77) find $\alpha_R^t[\varphi^A]$, consider the vacuum state ϕ_{λ_0}, cyclic for representation π, and compute the dynamics $\alpha_\pi^t[\varphi_\infty]$ dependent on the initial condition $\varphi_\infty(t=0) = \lambda$

$$\alpha_\pi^t[\varphi_\infty]^a\,[\lambda] = \lambda^a(t) = \lim_{R'\to\infty}\lim_{R\to\infty}\langle \phi_{\lambda_0},\, \beta_{R'}^\lambda \alpha_R^t[\varphi^A(x)]\rangle\,. \tag{79}$$

Unfortunately, this path can only be followed in its entirety in precisely solvable cases. Let us consider the equation of motion (77), remove the the truncation in the weak topology of the representation π associated with the vector ϕ_{λ_0} in it. In this case we have

$$\frac{d}{dt}\alpha_\pi^t[\varphi^i(x)] = F^i[\varphi^j(x), \lambda_0^A]\,. \tag{80}$$

The last equation has the property

$$\langle \phi_{\lambda_0},\, \frac{d}{dt}\alpha_\pi^t[\varphi^A(x)]\rangle = 0 \tag{81}$$

reflecting the stability $\pi:\ \ (\alpha_\pi^t)^*\phi_{\lambda_0} = \phi_{\lambda_0}$. For this the functional F must be special, for instance,

$$F^A[\varphi^i(x), \lambda_0^B] = F^A[\varphi^B(x) - \lambda_0^B]\,.$$

However, for $\phi_\lambda\ (\lambda \neq \lambda_0)$ we have

$$0 \neq \frac{d}{dt}\langle(\alpha_\pi^t)^*\phi_\lambda,\, \varphi^A(x)\rangle = \langle \phi_\lambda,\, F^A[\varphi^i(x), \lambda_0^B]\rangle\,. \tag{82}$$

A closed equation, obviously, arises if F^A is linear in $\{\varphi^B\}$ and does not depend on $\{\varphi^i,\ i \neq B\}$.

Often one can restrict oneself to the Hartree-Fock approximation. Let the dynamics has the form (80), then in this approximation

$$\frac{d}{dt}\alpha^t_\pi[\varphi^A_\infty] = F^A[\varphi^B_\infty, \lambda^B_0]\,. \tag{83}$$

We can generalise the whole construction by replacing the definition of the states (78) with the condition

$$\lim_{|\,\mathbf{x}\,|\to\infty} \langle \phi_\lambda,\, \varphi^A(x)\rangle = \lambda^A \tag{84}$$

for the case where the states $\lambda \neq \lambda_0$ are not translationally–invariant $(\alpha^{\mathbf{x}})^*\phi^\lambda \neq \phi^\lambda$. The right–hand side of Eq. (82) might simplify in the limit $|\ \mathbf{x}\ | \to \infty$ due to the properties of clustering (weakening of correlations) with respect to translations of the form

$$\lim_{|\,\mathbf{x}\,|\to\infty} \phi_\lambda(\alpha^{\mathbf{x}}[A]\,\alpha^{\mathbf{x}}[B]) = \lim_{|\,\mathbf{x}\,|\to\infty} \Big(\phi_\lambda(a^{\mathbf{x}}[A])\,\phi_\lambda(\alpha^{\mathbf{x}}[B])\Big)\,. \tag{85}$$

We now turn to the problem of the equilibrium states for systems with a non–trivial dynamics of variables at infinity. It is well known that the trace of the density matrix of translationally– invariant systems do not exist, because its logarithm is proportional to the space volume. Therefore, we first calculate the mean and other thermodynamic characterisation using local Gibbs states, ω^V_β,

$$\begin{aligned} &\omega^V_\beta(A) = \mathrm{Tr}(\rho^V_\beta\, A)\,, \qquad \rho^V_\beta = Z^{-1}_{\beta\,V}\, e^{-\beta\, H_V}\,, \\ &Z_{\beta\,V} = \mathrm{Tr} e^{-\beta\, H_V}\,. \end{aligned} \tag{86}$$

and then proceed to the thermodynamic limit, pushing V and, if necessary, other parameters to infinity. When V is finite the trace can be computed over various spaces of unitary–equivalent representations of the algebra of permutation relations, which at $V \to \infty$ become unitarily non–equivalent and give different answers.

The idea of the algebraic approach to the construction of the equilibrium state of a system consists in establishing a connection with its dynamics. If the dynamics is defined as the automorphisms α^t group of the algebra of observables $\mathcal{A}$, consider the equation of state $\omega_\beta \in \mathcal{A}^*$

$$\omega_\beta(\, A\,\alpha^t[B]\,)\Big|_{t=i\,\beta} = \omega_\beta(\, B\, A\,)\,, \qquad \forall\ A, B \in \mathcal{A}\,, \tag{87}$$

called the KMS (Kubo–Martin–Schwinger) condition. Note firstly that this condition is trivially satisfied for local Gibbs states (86). In known cases, when there is a reasonable way of calculating the thermodynamic limit and a well–defined limit state is obtained, it satisfies the KMS condition with

respect to the limit dynamics [110, 111]. In addition, the KMS states under sufficiently weak additional conditions, have a number of remarkable properties, allowing their decomposition into simpler states. The KMS states that do not admit further decomposition are called extreme and are interpreted as pure thermodynamic phases.

Thus, suppose that there exists a state $\langle ... \rangle_\beta \in \mathcal{M}^*$, satisfying the KMS condition on the dynamics of α^t as the of automorphisms of $\mathcal{M}$. For such a state, there is a theorem about central decomposition at infinity:

$$\langle w\, A \rangle_\beta = \int_{\aleph} \mu_\beta(d\lambda)\ (\lambda, w)_\beta\, \langle \lambda,\, A \rangle_\beta\,, \qquad \forall\, A \in \mathcal{A},\quad w \in \mathcal{Z}(\mathcal{A}'')\,. \tag{88}$$

It asserts the existence of a central measure μ_β concentrated on the set $\aleph$ of all KMS primitive states, performing a decomposition of the mean $\langle ... \rangle_\beta$ of any element of a bicommutant algebra of the form wA by the product of the average $\langle \lambda,\, A \rangle_\beta$ of $A \in \mathcal{A}$ over the near–margin KMS state with some functional on the centre of the bicommutant $(\lambda,\, w)$.

In any primary KMS state, the variables at infinity are fixed on their expectations. In order to satisfy the condition KMS, the measure μ_β must have certain properties of invariance. States $\langle \lambda,\, . \rangle_\beta$ as usual one can obtain via limits of the local Gibbs states:

$$\langle \lambda,\, A \rangle_\beta = \lim_{R \to \infty} \mathrm{Tr}_{\mathcal{H}_\lambda}(\rho_{\beta\, R} A), \tag{89}$$

where the trace is computed over a space characterised by a fixed value of the mean λ. A necessary condition for a state to be KMS is that it is t-invariant

$$\omega_\beta(\alpha^t[A]) = \omega_\beta(A)\,, \tag{90}$$

which must be satisfied. We have

$$\langle \lambda,\, \alpha^t[A] \rangle_\beta = \langle (\alpha^t_\pi)^* \lambda,\, A \rangle_\beta\,. \tag{91}$$

The measure μ_β must be arranged to compensate for the time dependence of states (91). The $(\lambda\,.\,)$ functional is normalised to unity and we are not interested in its explicit form while calculating the average of the local operators. If the dynamics of the variables at infinity $(\alpha^t_\pi)^*[\lambda_\infty]$ is Hamiltonian, then there exists a Liouville measure of their phase space, $(d\lambda)$, conserved by the Hamiltonian phase flow. Then states of the form

$$\omega_\beta(A) = \int (d\lambda)\, \langle \lambda,\, A \rangle_\beta \tag{92}$$

with $\langle \lambda,\, A\rangle_\beta$ defined using the thermodynamic limit of the local Gibbs states (89) are t-invariant. The last decomposition means that the equilibrium state is actually defined using the given representation of Π_S and the integral over $(d\lambda)$ accounts for the dynamics at infinity. Similarly, it can be demonstrated that the states (92) also satisfy directly the KMS condition.

4.2 Dynamics at infinity in the Yang–Mills theory

The evolution α_R^t for the Yang–Mills system in the Fock–Schwinger gauge is generated by the truncated Hamiltonian of the form

$$H_R = \frac{1}{2}\int d\mathbf{x}\, f_R(x)\,[\ \mathbf{E}_\perp^2 + E_\parallel^2 + \frac{1}{2}F_{ij}^2\ ]\,, \tag{93}$$

where $E_\parallel$ is given by the formula (7), and the smooth function $f_R(x)$ satisfies the properties (76). The canonical commutator is given by the formula

$$\begin{aligned}
&[\, A_\perp(g_1)\ ,\ E_\perp(g_2)\,] = i\,(g_1\ ,\ g_2)\,,\\
&(g_1\ ,\ g_2) = \int d\mathbf{x}\, g^a_{(1)\,i}(\mathbf{x})\, P_{ij}(\hat{\mathbf{x}})\, g^a_{(2)\,j}(\mathbf{x}), \quad g_{(A)} \in \mathcal{G}(\mathbf{R}^3)\ (A=1,2).
\end{aligned} \tag{94}$$

Since the surface terms are now zero due to the cut–off function, the equations of motion follow from (93) directly $\mathbf{A}_\perp(t) = \alpha_R^t[\mathbf{A}_\perp],\ \mathbf{E}_\perp(t) = \alpha_R^t[\mathbf{E}_\perp]$:

$$\dot{A}^a_{\perp\,k} = f_R\, E^a_{\perp\,k} - P_{kl}\nabla_l \tilde{\sigma}^a_R\,, \tag{95}$$

$$\dot{E}^a_{\perp\,k} = g t^{abc} E_k^{\perp\, b}\tilde{\sigma}^c_R - P_{kl}\nabla_i f_R\, F^a_{il}\,, \tag{96}$$

$$\tilde{\sigma}^a_R(\mathbf{x}) = \int_x^\infty dy\, f_R(y)\frac{1}{y^2}\int_0^y z^2\, dz\, \Phi^a_\perp(z\hat{\mathbf{x}})\,. \tag{97}$$

It is straightforward to verify the following commutation relation

$$[\, \Phi^a_\perp(\mathbf{x})\ ,\ \Phi^b_\perp(\mathbf{x}')\,] = -i\, g t^{abc}\Phi^c_\perp(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{x}')\,, \tag{98}$$

in deriving which we have used the identity $\nabla_i\nabla_j F_{ij} = 0$. Next we calculate the time derivative of $\Phi_\perp$

$$\begin{aligned}
\dot{\Phi}^a_\perp(\mathbf{x}) \;&=\; \frac{g}{2}t^{abc}[\Phi^b_\perp(\mathbf{x}), \int_0^\infty y^2 dy\, \Phi^c_\perp(y\hat{\mathbf{x}})]_+ \int_{\max(x,y)}^\infty \frac{dz}{z^2}\, f_R(z)\\
&+\; \frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2 f_R(x)\hat{x}_j\nabla_i F^a_{ij}\right).
\end{aligned} \tag{99}$$

Thus the integral with an arbitrary function of spherical angular coordinates ς will have zero time derivative

$$[\, H_R \,,\ \Phi_\perp(\varsigma)\,] = 0 \ \ \forall\, \varsigma(\hat{\mathbf{x}})\,, \qquad \Phi_\perp(\varsigma) \equiv \int d\mathbf{x}\, \Phi^a_\perp(\mathbf{x})\, \varsigma^a(\hat{\mathbf{x}}). \tag{100}$$

Therefore, $\Phi_\perp(\varsigma)$ generate the group G_{res} of residual gauge transformations β^ς, commuting with the cut–off dynamics $\alpha^t_R \beta^\varsigma = \beta^\varsigma \alpha^t_R$, and their action on the gauge field has the form

$$-i\,[\,\Phi_\perp(\varsigma)\ ,\ \mathbf{A}^a_\perp(\mathbf{x})\,] = \nabla\varsigma^a(\hat{\mathbf{x}}) \qquad \forall\ \varsigma^a(\hat{\mathbf{x}})\,, \tag{101}$$

From Eq. (99) after integrating by parts there follows the equation of motion for the variable $\tilde{\sigma}_R$

$$\begin{aligned} \dot{\tilde{\sigma}}^a_R(\mathbf{x}) &= -\frac{g}{2} t^{abc} \int_x^\infty dy\, [\tilde{\sigma}^{b\,\prime}_R(y\hat{\mathbf{x}}), \tilde{\sigma}^c_R(y\hat{\mathbf{x}})]_+ \\ &+ \int_x^\infty dy\, f^2_R(y)\, \hat{x}_j \nabla_i F^a_{ij}(y\hat{\mathbf{x}}). \end{aligned} \tag{102}$$

The quasi–local algebra $\mathcal{A}_l$ is generated by the fields $\mathbf{A}_\perp$, $\mathbf{E}_\perp$, but the equations of motion also contain a composite variable, which we rewrite as,

$$\tilde{\sigma}^a_R(\mathbf{x}) = f_R(\mathbf{x})\sigma^a(\mathbf{x}) - \sigma^a_R(\mathbf{x})\,, \quad \sigma^a_R = -\int_x^\infty dy\, f_R\,'(y)\sigma^a(y\hat{\mathbf{x}})\,, \tag{103}$$

$$\sigma^a = -(\,\Delta_x^{-1}\,\Phi^a\,)(\mathbf{x}) = \int_x^\infty \frac{dy}{y^2} \int_0^y z^2\, dz\, \Phi^a_\perp(z\hat{\mathbf{x}})\,. \tag{104}$$

In the $R \to \infty$ limit, the construction of σ^a_R will contain increasingly delocalised contributions due to the the carrier of the function supp $f'_R(y) = [R', R'']$. The representation of the algebra π_σ is fixed by the requirement that the field $\sigma(\mathbf{x})$ had an average value in it

$$\phi_\sigma(\sigma^a(\mathbf{x})) = \phi_\sigma(\alpha^t[\sigma^a(\mathbf{x})]) = \sigma^a$$

fixed in time. For $\mathbf{R}^3$-translation invariant states σ^a must be a constant.

Let us introduce a minimal set of $\mathcal{S}$-primary states over $\mathcal{A}_l$ containing $\phi_{\sigma\hat{\eta}}$ ($\sigma = \sigma\hat{\eta}$) and stable under $(\alpha^t_\pi)^*$, $(\beta^\varsigma)^*$. We denote as $\Pi_\mathcal{S}$ the representation of $\mathcal{A}_l$, given by the direct some of all representations from $\mathcal{S}$, and as $\mathcal{Z}_\mathcal{S}$ the center $\Pi_\mathcal{S}(\mathcal{A}''_l)$. Here

$$\mathcal{S} = \{\phi_{\sigma_\infty(\hat{\mathbf{x}})}\,,\ \forall\ \sigma_\infty(\hat{\mathbf{x}})\}$$

with an arbitrary function $\sigma_\infty(\hat{\mathbf{x}})$ taking values in $Lie - G_{res}$. The algebra $\mathcal{A}_l$ is presumed to be weakly asymptomatically Abelian with respect to translations in the representation π

$$w\text{-}\lim_{|\,\mathbf{x}\,|\to\infty} \pi\,(\,[\alpha^{\mathbf{x}}[A],\, B]\,) = 0\,, \qquad \forall\ A, B \in \mathcal{A}_l\,. \tag{105}$$

Then the variables at infinity

$$\sigma^a_\infty(\hat{\mathbf{x}}) = -w\text{-}\lim_{R\to\infty}\int_0^\infty dy\, {f_R}'\, \sigma^a(y\hat{\mathbf{x}})$$

exist in the weak topology $\Pi_{\mathcal{S}}$ for translationally invariant states ϕ:

$$\pi(\sigma_\infty) = \lim_{R\to\infty}\phi(\alpha_{R\hat{\mathbf{x}}}[\sigma]) = \phi(\sigma) = \sigma\,, \tag{106}$$

and also $\forall\ A \in \mathcal{A}_l$

$$[\,\sigma^a_\infty(\hat{\mathbf{x}})\ ,\ A\,] = -w\text{-}\lim_{R\to\infty}\int_0^\infty dy\, f'_R(y)\int_0^\infty \frac{z^2\,dz}{\max(y,z)}\,[\,\Phi^a_\perp(z\hat{\mathbf{x}})\ ,\ A\,] = 0\,. \tag{107}$$

The last relation is valid due to the finite region of integration by virtue of (105) and, in addition, by the property of the support supp ${f_R}'$. Hence, the variables at infinity belong to the centre of $\sigma_\infty(\hat{\mathbf{x}}) \in \mathcal{Z}_{\mathcal{S}}$. The symmetry β^ς commutes with the Hamiltonian and acts on the variables at infinity via the co–adjoint representation Ad^*

$$-i\,[\,\Phi_\perp(\varsigma)\ ,\ \sigma^a_\infty\,] = g t^{abc}\,\varsigma^b(\hat{\mathbf{x}})\,\sigma^c_\infty(\hat{\mathbf{x}})\,. \tag{108}$$

The effectively localised dynamics α^t_π of representation π has the form

$$\begin{aligned} \dot{A^a_{\perp\,k}} &= E^a_{\perp\,k} - P_{kl}\nabla_l(\,\sigma - \pi(\sigma_\infty)\,)^a\,, &(109)\\ \dot{E^a_{\perp\,k}} &= g t^{abc}\,E^b_{\perp\,k}\,(\,\sigma - \pi(\sigma_\infty)\,)^c - P_{kl}\nabla_i F^a_{il}\,, &(110)\end{aligned}$$

and from Eq (102) it follows that

$$\begin{aligned} \dot\sigma^a(\mathbf{x}) &= g t^{abc}\,\pi(\sigma_\infty)^b\sigma^c - \frac{g}{2}t^{abc}\int_x^\infty dy\,[{\sigma^b}'(y\hat{\mathbf{x}}), \sigma^c(y\hat{\mathbf{x}})]_+ \\ &+ \int_x^\infty dy\,\hat{x}_j\nabla_i F^a_{ij}(y\hat{\mathbf{x}}). \end{aligned} \tag{111}$$

However, note that from Eq. (99) there follows the equation of motion for the charge $Q^a_\infty(\hat{\mathbf{x}}) \equiv \int_0^\infty y^2 dy\,\Phi^a_\perp(y\hat{\mathbf{x}})$, which tells about the non–commutativity of the dynamics α^t_π with the symmetry transformations,

$$\dot{Q}^a_\infty(\hat{\mathbf{x}}) = g t^{abc} Q^b_\infty(\hat{\mathbf{x}})\,(\sigma^c_\infty - \pi(\sigma_\infty)^c). \tag{112}$$

We used the assumption that the variable at infinity, corresponding to the variable $x^2\hat{x}_j\nabla_i F_{ij}$, is zero, which is similar to condition (60) in the formalism with the boundary terms. Eq. (112) is the analogue of (64) and it is natural to require, the constraint, $Q_\infty = -\chi$, where χ is some additional variable at

infinity. This is achieved by a similar addition of a surface term of the form (46), where $\tilde{\sigma}_R$ plays the role of $A_0(\hat{\mathbf{x}})$. For any finite R such a term is zero, nevertheless the limiting expression in the weak topology is finite. These properties of the absence of weak continuity are typical for theories with long–range interactions and can be characterised as a *bifurcation* of an infinite dynamical system [98]. In the primary representation of π, the variable at infinity $\chi(\hat{\mathbf{x}})$ is fixed on its expectation, whose colour direction must coincide with $\pi(\sigma_\infty)$. The dynamics α^t_π at infinity is obtained from (112) by calculating the limit $x \to \infty$ averages over the states $\phi_{\vec{\sigma}_\infty(\hat{\mathbf{x}})}(\,.\,) = \langle \vec{\sigma}_\infty(\hat{\mathbf{x}}),\, .\,\rangle$ characterised by the property

$$\lim_{|\,\mathbf{x}\,|\to\infty} \phi_{\vec{\sigma}_\infty(\hat{\mathbf{x}})}(\vec{\sigma}(\mathbf{x})) = \vec{\sigma}_\infty(\hat{\mathbf{x}}) \tag{113}$$

In the Hartree–Fock approximation, meaning the condition of clustering (weakening of correlations) of the form (85) and this is zero. We finally write

$$\dot{\sigma}^a_\infty(\hat{\mathbf{x}}) = -g t^{abc}\, \sigma^b_\infty(\hat{\mathbf{x}})\, \pi(\sigma_\infty)^c\,, \quad \sigma^a_\infty(\hat{\mathbf{x}}) \in \mathcal{Z}_{\mathcal{S}}, \tag{114}$$

which is the equation of colour rotations around the vector $\pi(\sigma_\infty)$.

To conclude this section, we emphasise that the formalism of variables at infinity leads to the same physical conclusions as the Hamiltonian formalism with surface terms in the infinite limit. For this reason, we shall hereafter freely use both terms "variables at infinity" and "variables at the boundary" as equivalent, implying that they appear in somewhat different mathematical contexts.

4.3 Poincaré algebra in the Fock–Schwinger gauge

The obvious shortcomings of the Fock–Schwinger gauge are the lack of explicit Poincaré- and even $\mathbf{R}^3$- translational covariance. This in turn complicates the renormalisation procedure. Nevertheless, the theory still possesses space–time symmetries, though this fact is not as obvious as in the covariant gauges. The existence of these symmetries of the Yang–Mills system imply the existence of operators $Q^A_R \in \mathcal{A}_l$ generating symmetry transformations

$$\alpha^q[A] = w\text{-}\lim_{R\to\infty} e^{i\,Q_R q}\, A\, e^{-i\,Q_R q}\,, \qquad A \in \mathcal{A}_l\,.$$

At the formal level, the existence of the Poincaré algebra in physical gauge was demonstrated in Ref. [112]. There it was shown that the operators

$$P_0 = H = \frac{1}{2}\int d\mathbf{x}\,(\mathbf{E}^2 + \mathbf{B}^2)\,, \qquad P_i = \int d\mathbf{x}\, T_{0i}\,,$$

$$M_{ij} = \int d\mathbf{x}\,(x_i T_{0j} - x_j T_{0i} + E_i^a A_j^a - E_j^a A_i^a)\,,$$
$$M_{0i} = t\,P_i - \frac{1}{2}\int d\mathbf{x}\; x_i\,(\mathbf{E}^2 + \mathbf{B}^2)\,, \qquad T_{0i} = -\mathbf{E}^a \partial_i \mathbf{A}^a, \tag{115}$$

restricted on the surface of the constraint and the gauge condition, satisfy the commutation relations of the Poincaré group.

The presence of variables at infinity in the FS gauge leads to additional complexities. Therefore, it is necessary to make sure that the translational $\mathbf{R}^3$ invariance still exists, since its presence is necessary for a consistent formulation of the theory. Note that in the Fock–Schwinger gauge one can write

$$P_i = -\int d\mathbf{x}\,\mathbf{E}_\perp^a \partial_i \mathbf{A}_\perp^a\,, \tag{116}$$

and $P_0 = H$, where H is the Hamiltonian of the system.

In order to make sense of the P_μ operators we need to introduce infrared regularisation by means of the function $f_R(x)$. For functions $f_R(x)$ depending only on the radius it turns out that the commutation relations will be fulfilled in the weak sense

$$\lim_{R,\,R'\to\infty} \phi(\,[P_\mu^R, P_\nu^{R'}]\,) = 0\,, \qquad \forall\,\phi \in \pi\,.$$

For the commutator of the spatial components of the momentum with each other this formula is trivially verified. It is much more interesting to consider the commutator $[H, P_i]$. We proceed with the QED case, where the variables at infinity are absent in the equations of motion. As pointed out in Chapter 2, physical quantities in the FS gauge can be rewritten in terms of the diagonalising Coulomb variables. In this case, the algebraic properties (30,31) of the transition operators are used and the integration by parts is performed, which is admissible due to the presence of a spherically symmetric cut–off function. It turns out that in the limit of the removed cut–off

$$P_i^{FS} = P_i^{Cuol}\,,$$

as this happens for the Hamiltonian (3).

We now turn to the case of Yang–Mills fields. We write the expectation of the commutator as

$$\lim_{R\,R'} i\,\phi([H_{R'}, P_i^R]) = \lim_{R\,R'} \frac{d}{dt}\bigg|_{t=0} \phi(\alpha_{R'}^t[P_i^R]) = \lim_{R} \frac{d}{dt}\bigg|_{t=0} \phi(\alpha_\pi^t[P_i^R])\,,$$

$P_i^R \in \mathcal{A}_l$. All we need is the derivative $\frac{d}{dt}\alpha_\pi^t$, which we calculate using (109,110). It consists of of the local part and the contribution containing $\pi(\sigma_\infty)$. After a simple but somewhat long transformations,

the first summand can be written as the integral of the gradient of a scalar even function, which is therefore nullified. The remaining construction

$$\delta\left(\frac{d}{dt}\alpha_\pi^t[\mathbf{E}^a\partial_i\mathbf{A}^a]\right) = E_k^a\nabla_k\partial_i\,\pi(\sigma_\infty)\,, \tag{117}$$

is obviously zero if and only if $\pi(\sigma_\infty)$ is a constant. This is the necessary and sufficient condition for the dynamics, which is representation–dependent, to preserve the translational invariance $\alpha_\pi^t\alpha^{\mathbf{x}} = \alpha^{\mathbf{x}}\alpha_\pi^t$ in the considered representation π.

5 Dependence of the partition function on the variables at the boundary

In this Chapter, we study the dependence of the partition function of the Abelian and non–Abelian gauge theories on the value of the variable at the boundary $|\chi(\hat{\mathbf{x}})|$ following our work [113]. In the framework of the Hamiltonian formalism of Chapter 3 we have established that the surface Lagrange multiplier $A_0(R\hat{\mathbf{x}})$ is not determined from the equations of motion. The functional integral includes all its possible values. The action of the system without the surface term is invariant with respect to time–dependent gauge transformations. When taking into account the non–zero surface term with $|\chi|$, this invariance is absent. The direction of the vector χ, as we have shown, must be treated as a Hamiltonian variable and has nontrivial (64) dynamics, while its length is fixed and plays the role of an external parameter characterising different phases of the system.

In the theory at finite temperature, the possibility of time–dependent gauge transformations is furthermore limited by the the requirement of preservation of the periodic boundary conditions in time for the gauge fields, which will be discussed in Chapter 6. The question about physically feasible values of $|\chi|$ at a given finite temperature is reduced to finding the most statistically probable values at which the partition function is maximal, and to the vacuum values at which the energy is minimal for zero temperature.

5.1 Electrodynamics with an external charge

We shall begin our study with a simple case that admits an explicit solution, namely the electrodynamics with an external charge density. We can work in the Coulomb gauge here. Note that from the

relations (33) and (35) there follows the expression of $\hat{\mathbf{x}}E^{\|}(R\hat{\mathbf{x}})$ in the Gauss law through the physical transverse variables of the Coulomb gauge.

$$R^2\hat{\mathbf{x}}E^{\|}(R\hat{\mathbf{x}}) - \int_0^R y^2 dy\, \rho(y\hat{\mathbf{x}}) = \hat{\Delta}\int_0^R (R-y)dy\, \hat{\mathbf{x}}\mathbf{E}^{\perp}(y\hat{\mathbf{x}})\,, \tag{118}$$

where we used the notation for the spherical part of the Laplacian $\Delta = x^{-1}\partial_x^2 x + x^{-2}\hat{\Delta}$.

The partition function of the theory in the spherical domain can be represented as the following functional integral

$$\begin{aligned} Z = &\int \mathcal{D}\mathbf{A}\mathcal{D}\mathbf{E}\,\delta(\partial\mathbf{A})\delta(R^2E_{\|}(R\hat{\mathbf{x}}) + \chi(\hat{\mathbf{x}})) \\ &\exp\int_\Lambda d^4x \left(i\mathbf{E}\dot{\mathbf{A}} - \frac{1}{2}\mathbf{E}^2 + \frac{1}{2}\mathbf{A}\Delta\mathbf{A} - \frac{1}{2\epsilon^2}(\partial\mathbf{E}-\rho)^2\right), \end{aligned} \tag{119}$$

where we introduced the notation $\Lambda = [0,\beta]\times V$ (we also use the notation $\partial\Lambda = [0,\beta]\times\partial V$.) The Gaussian constraint is regularised by using a small parameter ϵ, which must be set to zero at the end of calculations. This Gaussian integral is computed in the standard way by shifting the integration variables. To calculate the integral over $\mathbf{E}$, we introduce a new integration variable, $\mathbf{E}_1$,

$$\mathbf{E} = \mathbf{E}_1 + \mathcal{E}\,, \qquad \mathbf{E}_1(R\hat{\mathbf{x}}) = 0\,. \tag{120}$$

Here, the new variable $\mathbf{E}_1$ satisfies the zero boundary condition and $\mathcal{E}$ is chosen so that there is no linear term on $\mathbf{E}_1$. This gives the following equation for determining $\mathcal{E}$

$$i\dot{\mathbf{A}} - \mathcal{E} + \frac{1}{\epsilon^2}\partial(\partial\mathcal{E}) - \frac{1}{\epsilon^2}\partial\rho = 0\,, \tag{121}$$

$$R^2\hat{\mathbf{x}}\mathcal{E}(R\hat{\mathbf{x}}) + \chi(\hat{\mathbf{x}}) = 0\,. \tag{122}$$

The last boundary condition follows from the second delta function in (119). It is possible to decompose this vector into longitudinal and transverse components in the momentum space $\mathcal{E} = \mathcal{E}^{\perp} - \partial\varphi$. Then the transverse component is $\mathcal{E}^{\perp} = i\dot{\mathbf{A}}$, and the equation for φ takes the form

$$(\Delta - \epsilon^2)\varphi = -\rho\,, \tag{123}$$

$$R^2\frac{\partial\varphi}{\partial R} = \chi(\hat{\mathbf{x}})\,. \tag{124}$$

The partition function (119) can be represented as a product

$$\begin{aligned} &Z = Z_1\tilde{Z}\,, \\ &Z_1 = \int \mathcal{D}\mathbf{A}^{\perp}\mathcal{D}\mathbf{E}\,\exp\int_\Lambda d^4x\,\left(-\frac{1}{2}\dot{\mathbf{A}}^2 + \frac{1}{2}\mathbf{A}\Delta\mathbf{A} - \frac{1}{2}\mathbf{E}_1^2 - \frac{1}{2\epsilon^2}(\partial\mathbf{E}_1)^2\right), \\ &\tilde{Z} = \exp\beta\left(-\frac{1}{2}\int_{\partial V} d\hat{\mathbf{x}}\,\chi(\hat{\mathbf{x}})\varphi(R\hat{\mathbf{x}}) - \frac{1}{2}\int_{\partial V} d\mathbf{x}\,\rho(\mathbf{x})\varphi(\mathbf{x})\right), \end{aligned} \tag{125}$$

where φ is the solution of (123,124).

The solution (123,124) is obviously the sum of the homogeneous part, ϕ satisfying the nontrivial boundary condition, and the inhomogeneous part satisfying the zero boundary condition

$$\varphi = \phi - G \bullet \rho \qquad G = (\Delta - \epsilon^2)^{-1}\,, \tag{126}$$

where G is the Green's function with the Neumann zero boundary condition. We introduce the following notations

$$\begin{aligned}
\tilde{Z} &= \tilde{Z}_\chi \tilde{Z}_{\rho\rho} \tilde{Z}_{\rho\chi}\,, \\
\tilde{Z}_\chi &= \exp\left(-\frac{\beta}{2}\int_{\partial V} d\hat{\mathbf{x}}\,\chi(\hat{\mathbf{x}})\phi(\hat{\mathbf{x}})\right)\,, \\
\tilde{Z}_{\rho\rho} &= \exp\left(\frac{\beta}{2}\int_{\partial V} d\mathbf{x}d\mathbf{y}\,\rho(\mathbf{x})G(\mathbf{x},\mathbf{y})\rho(\mathbf{y})\right)\,, \\
\tilde{Z}_{\rho\chi} &= \exp\left(-\frac{\beta}{2}\int_{\partial V} d\mathbf{x}\,\phi(\mathbf{x})\rho(\mathbf{x})\right)\exp\left(\frac{\beta}{2}\int_{\partial V} d\hat{\mathbf{x}}\,\chi(\hat{\mathbf{x}})\,(G\bullet\rho)(\hat{\mathbf{x}})\right)\,.
\end{aligned} \tag{127}$$

It is natural to consider this problem in terms of spherical coordinates. The solution of (123) regular inside the sphere is given by the formula

$$\phi_{lm} = C_{lm}\sqrt{\frac{\pi}{2\epsilon r}} I_{l+1/2}(\epsilon r) \tag{128}$$

$$I_{n-1/2}(z) = \sqrt{\frac{2}{\pi z}} z^n \left(\frac{1}{z}\frac{d}{dz}\right)^n \cosh z\,. \tag{129}$$

The constant C_{lm} is thus defined by equation (124). The role of the regulariser now ϵ becomes clear. The solution for the zero mode will be

$$\phi_{00} = C_{00}\frac{\sinh\epsilon r}{\epsilon r}\,, \qquad C_{00} = \frac{\chi_{00}}{R^2\epsilon(\cosh\epsilon R/\epsilon R - \sinh\epsilon R/(\epsilon R)^2)}\,, \tag{130}$$

and it has a $1/\epsilon^2$ singularity when ϵ tends to zero. At the same time, the solutions for the other modes are regular in this limit and tend to

$$\phi_{lm} = C_{lm}\, r^l\,, \qquad C_{lm} = \frac{\chi_{lm}}{lR^{l+1}}\,. \tag{131}$$

In the consideration of the zero mode one should be very careful and hold a finite ϵ. The Green's function for the zero mode is defined as a solution to the equations

$$\left(\frac{1}{r}\frac{\partial^2}{\partial r^2} r - \epsilon^2\right) G_{00}(r,r') = \frac{\delta(r-r')}{rr'}\,, \tag{132}$$

$$\left.\frac{\partial G_{00}(r,r')}{\partial r}\right|_{r=R} = 0 \tag{133}$$

and can be easily found in the explicit form

$$G_{00}(r,r') = \frac{1}{\epsilon r r'}\left(\frac{1}{2}\sinh\epsilon|r-r'| - \frac{1}{2}\sinh\epsilon(r+r') + \frac{\sinh\epsilon R - \cosh\epsilon R/\epsilon R}{\cosh\epsilon R - \sinh\epsilon R/\epsilon R}\sinh\epsilon r\sinh\epsilon r'\right). \quad (134)$$

The leading terms in the Laurent decomposition at small ϵ have the form

$$G_{00}(r,r') \simeq -\frac{3}{\epsilon^2 R^3} + \frac{9}{5R} - \frac{1}{2}\frac{r^2+r'^2}{R^3} - \frac{1}{\max(r,r')}, \quad (135)$$

$$\phi_{00} \simeq \left(\frac{3}{\epsilon^2 R^3} - \frac{3}{10R} + \frac{r^2}{2R^3}\right)\chi_{00}. \quad (136)$$

Substituting these expressions into the formulae (127) gives for the non–zero modes in the limit $\epsilon = 0$

$$\tilde{Z}_{\chi\, l>0} = \exp\left(-\frac{\beta}{2R}\sum_{lm>0}\frac{|\chi_{lm}|^2}{l}\right). \quad (137)$$

For simplicity, we assume that the charge distribution ρ is spherically symmetric. In this case it gives a contribution only to the zero mode. We also introduce the notations for the integrals

$$\mathcal{Q}_{00} = \int_0^R r^2\, dr\, \rho_{00}(r), \quad (138)$$

$$\mathcal{G}_{00} = \int_0^R r^4\, dr\, \rho_{00}(r). \quad (139)$$

Then the result can be represented as

$$\tilde{Z}_{00} = \exp\beta\Big(-\frac{1}{2}\int_0^R x^2dx\, y^2dy\,\frac{\rho_{00}(x)\,\rho_{00}(y)}{\max(x,y)} - \frac{3}{2\epsilon^2R^3}(\mathcal{Q}_{00}+\chi_{00})^2 + \frac{1}{R}(\frac{9}{10}\mathcal{Q}_{00}^2 - \frac{1}{10}\chi_{00}^2 + \frac{3}{10}\chi_{00}\mathcal{Q}_{00}) - \frac{1}{2R^3}(\mathcal{Q}_{00}+\chi_{00})\mathcal{G}_{00}\Big). \quad (140)$$

In the limit $\epsilon \to 0$, this expression will contain the delta function of the condition $\mathcal{Q}_{00} + \chi_{00} = 0$, and we arrive at the following correction to the standard answer

$$\tilde{Z}_{00} = \exp\left(\frac{\beta}{2R}\mathcal{Q}_{00}^2\right)\,\delta(\mathcal{Q}_{00}+\chi_{00}). \quad (141)$$

As a simple illustration of this result, consider the charge density of charge $\rho_{00} = -\kappa/(\sqrt{\pi}\,r)$ corresponding to the linearly increasing potential $\varphi = \kappa\, r$. For this exotic charge distribution, modelling the confinement potential, we obtain an additional constant contribution to the free energy density

$$\Delta\mathcal{F} = -\frac{\log Z}{\beta V} = -\frac{3}{32\pi^2}\kappa^2. \quad (142)$$

It is interesting to emphasise that its occurrence is entirely due to the boundary terms, and this addition reduces the free energy. This example suggests a possible connection between the confinement phenomenon in the non–Abelian gauge theory and boundary effects.

5.2 Formulation in terms of the collective variables

Here I am going to consider a slightly different formulation of this problem by means of introducing a collective variable, σ, conjugate to the Gaussian constraint. Both formulations are equivalent in the Abelian theory, where the transition from one to the other is carried out by a trivial change of the integration variables. The new formulation turns out to be more fruitful in the non–Abelian theory. We can rewrite the formula (119) in terms of the collective variable σ, introduced by means of the definition

$$\exp\left(-\frac{1}{2\epsilon^2}\int_\Lambda d^4x\,(\partial\mathbf{E}-\rho)^2\right)=\int\mathcal{D}\sigma\,\exp\int_\Lambda d^4x\,\left(-\frac{\epsilon^2}{2}\sigma^2+i\sigma(\partial\mathbf{E}-\rho)\right). \tag{143}$$

We calculate the integral over $\mathbf{E}$

$$\begin{aligned}&\int\mathcal{D}\mathbf{E}\,\exp\int_\Lambda d^4x\,\left(-\frac{1}{2}\mathbf{E}^2+i\mathbf{E}\dot{\mathbf{A}}+i\sigma\partial\mathbf{E}\right)\,\delta(R^2E_\parallel(R\hat{\mathbf{x}})+\chi(\hat{\mathbf{x}}))\\&=\exp\left(-\frac{1}{2}\int_\Lambda d^4x(\dot{\mathbf{A}}^2+(\partial\sigma)^2)-i\int_{\partial\Lambda}dt\,d\hat{\mathbf{x}}\,\chi\sigma\right).\end{aligned} \tag{144}$$

After integration by parts in the summand $\int d\mathbf{x}\,\sigma\partial\mathbf{E}$ we have applied the condition that follows from the delta function in the surface term. In addition, we have performed integration by parts in the summand $\int d\mathbf{x}\,\dot{\mathbf{A}}\partial\sigma$, where the corresponding surface term vanishes due to the gauge condition. The dependence on the variable χ is contained only in the integral

$$\tilde{Z}=\int\mathcal{D}\sigma\,\exp\left(-\int_\Lambda d^4x\,(\frac{1}{2}(\partial\sigma)^2+\frac{\epsilon^2}{2}\sigma^2+i\sigma\rho)-i\int_{\partial\Lambda}dt\,d\hat{\mathbf{x}}\,\chi\sigma\right). \tag{145}$$

It can be seen that the boundary condition on $E_\parallel$ is equivalent to an additional surface term in this formula. The integral is calculated directly by shifting the integration variable $\sigma=\sigma_1+\varsigma$. The new variable satisfies the boundary condition $R^2\sigma_1'(R\hat{\mathbf{x}})=0$, and ς is found from the condition of of absence of the linear term in σ_1 after this shifting. This gives the equation on ς

$$(\Delta-\epsilon^2)\varsigma=i\rho\,, \tag{146}$$

from which we obtain $\tilde{Z} = \tilde{Z}_1 \tilde{Z}_2$,

$$\begin{aligned}\tilde{Z}_1 &= \int \mathcal{D}\sigma_1 \, \exp\Big(- \int_\Lambda d^4x \, (\frac{1}{2}(\partial\sigma_1)^2 + \frac{1}{2}\epsilon^2\sigma_1^2) \\ &- \int_{\partial\Lambda} dt \, d\hat{\mathbf{x}} \, \sigma_1(R^2\varsigma' + i\chi)\Big), \qquad (147) \\ \tilde{Z}_2 &= \exp\left(-\frac{i}{2}\int_\Lambda d^4x \, \rho\,\varsigma - \int_{\partial\Lambda} dt \, d\hat{\mathbf{x}} \, \varsigma(\frac{1}{2}R^2\varsigma' + i\chi)\right). \qquad (148)\end{aligned}$$

Note that the boundary condition on σ_1 allows a certain arbitrariness, namely $\sigma_1 = \sigma_2 + \sigma_R(\hat{\mathbf{x}})$, $\sigma_2(R\hat{\mathbf{x}}) = 0$. Integration over *variables at the boundary* $\sigma_R(\hat{\mathbf{x}})$ gives a surface delta function

$$\int \mathcal{D}\sigma_R(\hat{\mathbf{x}}) \exp\left(-\int_{\partial\Lambda} dt \, d\hat{\mathbf{x}} \, \sigma_R(\hat{\mathbf{x}})(R^2\varsigma' + i\chi)\right) = \delta(R^2\varsigma' + i\chi)\,. \qquad (149)$$

Applying the boundary condition ς, we finally obtain

$$\tilde{Z} = \exp\left(-\frac{i}{2}\beta\int_V d\mathbf{x}\,\varsigma\rho - \frac{i}{2}\beta\int_V d\hat{\mathbf{x}}\,\varsigma\,\chi\right). \qquad (150)$$

By renaming $\varsigma = -i\varphi$, we completely reproduce the answer (123,124,125) obtained earlier. Thus, the formulation in terms of the collective variable is equivalent to the traditional if we take into account the contribution of surface terms and the integration over the variable at the boundary $\sigma_R(\hat{\mathbf{x}})$.

5.3 Effective action of gluodynamics

Let us consider the Faddeev–Popov functional integral for the generating functional of the Green's functions [95, 96]

$$\begin{aligned}Z[\chi;\zeta,\eta] &= \int \mathcal{D}\mathbf{A}_\perp \, \mathcal{D}\mathbf{E}_\perp \, \mathcal{D}A_0(R\hat{\mathbf{x}}) \, \exp\Big(- \int_\Lambda d^4x \, [\mathbf{E}\dot{\mathbf{A}} - \frac{1}{2}\mathbf{E}^2 + \frac{1}{2}\mathbf{B}^2 \\ &+ i\zeta E_\parallel - \eta B_\parallel] + \int_{\partial\Lambda} dt \, d\hat{\mathbf{x}} \, A_0(R\hat{\mathbf{x}}) \, (R^2 E_\parallel(R\hat{\mathbf{x}}) + \chi(\hat{\mathbf{x}}))\Big). \qquad (151)\end{aligned}$$

In this formula, it is assumed that $E_\parallel$ is expressed in terms of the transverse components using the relation (7). We shall use the Euclidean notation here, in which the electric field is purely imaginary. The fields satisfy the periodic boundary conditions in time on the interval $[0, \beta]$ and the boundary conditions (58-62) at spatial infinity. We have introduced sources for the longitudinal components of the fields and the surface term with χ. For the sake of brevity, the source terms below will generally not be written out explicitly if not necessary.

We can use the following simple observation. In the expressions

$$\mathbf{E}^2 = \mathbf{E}_\perp^2 + E_\parallel^2\,, \qquad \mathbf{B}^2 = \mathbf{B}_\perp^2 + B_\parallel^2\,, \qquad \mathbf{B}_\perp = P\,\mathbf{B} = P\,\mathrm{rot}\,\mathbf{A}_\perp$$

it can be seen that all nonlocal and non–Abelian constructions are contained only in the longitudinal components. It is useful to introduce two auxiliary periodic in time real variables, λ and ν by means of the integral representations

$$\exp\,(\frac{1}{2}\int_\Lambda dx\,E_\parallel^2\,) \;=\; \int \mathcal{D}\lambda\;\exp\,(\int_\Lambda dx\,[-\frac{1}{2}\lambda^2 + \lambda E_\parallel\,]\,)\,, \tag{152}$$

$$\exp\,(-\frac{1}{2}\int_\Lambda dx\,B_\parallel^2\,) \;=\; \int \mathcal{D}\,\nu\;\exp\,(\int_\Lambda dx\,[-\frac{1}{2}\nu^2 + i\,\nu B_\parallel\,]\,)\,. \tag{153}$$

We perform a transformation

$$\int_{V_R} d\mathbf{x}\,\lambda(x\hat{\mathbf{x}})\frac{1}{x^2}\int_0^x y^2\,dy\,\Phi_\perp(y\hat{\mathbf{x}}) = \int_{V_R} d\mathbf{x}\,\Phi_\perp(\mathbf{x})\,\tilde{\sigma}_R(\mathbf{x})\,, \tag{154}$$

$$\tilde{\sigma}_R(\mathbf{x}) \equiv \int_x^R dy\,\lambda(y\hat{\mathbf{x}}) \tag{155}$$

and then integrate by parts

$$\begin{aligned}
&\int_{V_R} d\mathbf{x}\;\tilde{\sigma}_R(\mathbf{x})\,\partial\mathbf{E}_\perp = -\int_{V_R} d\mathbf{x}\;\mathbf{E}_\perp\,P\partial\,\tilde{\sigma}_R + \int_{\partial V_R} d\hat{\mathbf{x}}\;R^2\mathbf{E}_\perp(R\hat{\mathbf{x}})\,\tilde{\sigma}_R(R\hat{\mathbf{x}}),\\
&\int_{V_R} d\mathbf{x}\;\nu\,\hat{\mathbf{x}}\mathrm{rot}\,\mathbf{A}_\perp = \int_{V_R} d\mathbf{x}\;\mathbf{A}_\perp\;[\,\partial\,\nu\;,\;\hat{\mathbf{x}}\,] + \int_{\partial V_R} d\hat{\mathbf{x}}\;R^2\nu\;[\,\hat{\mathbf{x}}\;,\;\mathbf{A}_\perp\,]\,.
\end{aligned} \tag{156}$$

The surface terms are identically zero due to the definition of the of transversality. Note that the integrals over the transverse components become Gaussian after introduction of the collective variables and can be explicitly computed. For a more compact notation it is convenient to use a new variable similar to the one we introduced in the formula (72), and taking non–trivial values at the boundary,

$$\sigma(\mathbf{x}) \equiv A_0(R\hat{\mathbf{x}}) + \tilde{\sigma}_R(\mathbf{x}), \tag{157}$$

and by integration over this variable we shall understand the integrations over the original variables $\mathcal{D}\sigma \equiv \mathcal{D}\lambda\,\mathcal{D}A_0(R\hat{\mathbf{x}})$. Next, we integrate over $\mathbf{E}_\perp$

$$\begin{aligned}
I \;&=\; \int \mathcal{D}\mathbf{E}_\perp\;\exp\,(\int_\Lambda dx\,[\,\frac{1}{2}(\mathbf{E}^a)^2 - \mathbf{E}^a\,(\,\dot{\mathbf{A}}^a + P\partial\,\sigma^a - g t^{abc}\,\mathbf{A}^b\,\sigma^c\,)\,]\,)\\
&=\; \exp\,(-\frac{1}{2}\int_\Lambda dx\,[\,\dot{\mathbf{A}}^a + P\partial\,\sigma^a - g t^{abc}\,\mathbf{A}^b\,\sigma^c\,]^2\,)\,.
\end{aligned} \tag{158}$$

In the integral over $\mathbf{A}_\perp$, which arises after integration over the electric field, we make a rotation

$$A_{\perp\, i} = \varepsilon_{ijk}\, \mathcal{A}_{\perp\, j} \hat{x}_k,$$

preserving the integration region and having as the Jacobian a non–zero normalisation constant. In the integral
$\int_{V_R} d\mathbf{x}\, \left(\frac{1}{x}\hat{\mathbf{x}}\partial(x\mathcal{A}_\perp)\right)^2$ it is possible to integrate by parts, and the surface term is zero due to the boundary condition (62). After these calculations, the partition function is written as

$$\begin{aligned} Z &= \int \mathcal{D}\,\mathcal{A}_\perp\, \mathcal{D}\,\sigma\, \mathcal{D}\,\nu\ \exp\Big(\, -\frac{1}{2}\nu \bullet \nu - \frac{1}{2}\partial\sigma \bullet \partial\sigma \\ &- \frac{1}{2}\mathcal{A} \bullet M \bullet \mathcal{A} + \mathcal{A} \bullet N\, \Big)\,, \qquad (159) \\ M^{ab} &= (-\Delta_x\, \delta^{ab} - (\nabla_t^2)^{ab})\, P + i\, g t^{abc}\, \nu^c\, L\,, \qquad (160) \\ N_i^a &= i\, P_{ij}\, \partial_j\, \nu^a - L_{ij}\, \nabla_t^{ab}\, \partial_j\, \sigma^b\,, \qquad \nabla_t^{ab} = \delta^{ab}\, \partial_t - g t^{abc}\, \sigma^c\,. \qquad (161) \end{aligned}$$

Here the bullet $\bullet$ means integration over the domain Λ, and operators P and L are defined as

$$L_{ij} = \varepsilon_{ijk}\, \hat{x}_k, \qquad P_{ij} = \delta_{ij} - \hat{x}_i\, \hat{x}_j. \qquad (162)$$

It is trivially verified that they satisfy a simple algebra

$$P^2 = P, \qquad L^2 = -P, \qquad LP = PL = L. \qquad (163)$$

Let α, β and γ be two arbitrary operators, which commute with P and L, but not necessarily with each other. We denote the full trace with a capital letter, and the trace over the degrees of of freedom except those corresponding to P and L with a lowercase one. The algebra (163) allows one to derive

$$\begin{aligned} (\,\alpha\, P + \beta\, L\,)^n &= \alpha_n\, P + \beta_n\, L\,, \\ \alpha_n &= \mathrm{Re}\,(\alpha + i\,\beta)^n \equiv \frac{1}{2}[\,(\alpha + i\,\beta)^n + (\alpha - i\,\beta)^n\,]\,, \\ \beta_n &= \mathrm{Im}\,(\alpha + i\,\beta)^n \equiv \frac{1}{2\,i}[\,(\alpha + i\,\beta)^n - (\alpha - i\,\beta)^n\,]\,, \end{aligned}$$

which gives

$$\begin{aligned} \mathrm{Tr}\log[\,\gamma\,\mathbf{1} + \alpha\, P + \beta\, L\,] &= \mathrm{tr}\log\gamma + \mathrm{tr}\log[\,(\alpha + \gamma + i\,\beta)\,(\alpha + \gamma - i\,\beta)\,]\,, \\ (\,\gamma\,\mathbf{1} + \alpha\, P + \beta\, L\,)^{-1} &= \gamma^{-1}(\mathbf{1} - P) + \mathrm{Re}\,(\,\alpha + \gamma + i\,\beta)^{-1}\, P + \\ &+ \mathrm{Im}\,(\,\alpha + \gamma + i\,\beta)^{-1}\, L\,. \qquad (164) \end{aligned}$$

The integral over $\mathcal{A}_\perp$ is calculated directly and with the use of the above formulae one can exclude the spatial vector structure from the result. A slightly different and more elegant way to treat this is to introduce the projection operators and the corresponding components of the transverse vectors

$$\Pi_\pm = \frac{1}{2}(P \pm iL)\,, \quad \mathbf{a}_\pm = \Pi_\pm\,\mathbf{a}. \tag{165}$$

They also possess a simple algebra

$$\Pi_\pm = \Pi_\pm^2\,, \quad \Pi_+\Pi_- = 0\,, \quad \Pi_+ + \Pi_- = P\,, \tag{166}$$

We can rewrite (159) via $\mathcal{A}_\pm$ projections

$$Z = \int \mathcal{D}\,\mathcal{A}_+\,\mathcal{D}\,\mathcal{A}_-\,\mathcal{D}\,\sigma\,\mathcal{D}\,\nu\ \exp\left(-W[\sigma,\nu,\mathcal{A}_\pm]\right), \tag{167}$$

$$W[\sigma,\nu,\mathcal{A}_\pm] = \frac{1}{2}\nu\bullet\nu + \frac{1}{2}\partial\sigma\bullet\partial\sigma + \frac{1}{2}\mathcal{A}\bullet\mathcal{C}\bullet\mathcal{A} + i\,\mathcal{A}\bullet\mathcal{K}\,,$$

$$\mathcal{A} = (\,\mathcal{A}_-\,,\,\mathcal{A}_+\,)\,, \qquad \mathcal{K} = (\,K_+\,,K_-\,)\,,$$

$$\mathcal{C} = \begin{pmatrix} 0 & C_+ \\ C_- & 0 \end{pmatrix}, \qquad D^{ab} = gt^{abc}\,\nu^c\,,$$

$$C_\pm = -\Delta_x - \nabla_t^2 \pm D\,, \qquad K_\pm = \partial_\pm\,\nu \pm \nabla_t\,\partial_\pm\,\sigma\,.$$

The integrals over $\mathcal{A}_\pm$ are now elementary. We have

$$\begin{aligned} Z[\chi] &= \int \mathcal{D}\,\sigma\,\mathcal{D}\,\nu\ \exp\left(-W[\sigma,\nu] + \int_{\partial\Lambda} dt\,d\hat{\mathbf{x}}\,\sigma\chi\right), \\ W[\sigma,\nu] &= \frac{1}{2}\nu\bullet\nu + \frac{1}{2}\partial\sigma\bullet\partial\sigma + \frac{1}{2}K_-\bullet C_+^{-1}\bullet K_+ \\ &+ \frac{1}{2}K_+\bullet C_-^{-1}\bullet K_- + \frac{1}{2}\mathrm{tr}\log C_+C_-\,, \end{aligned} \tag{168}$$

where we have explicitly written out the dependence on the parameter χ. Allow me to discuss the issue of accounting for the sources. Firstly, in order to include the source J of the gauge field $\mathbf{A}_\perp$ one must add to the action the term $\Delta W = gJ_0\bullet\sigma$ and change $K_\pm \to K_\pm \pm gJ_\pm$ in the previous formula.

We can relate the generating functional of the longitudinal components of the strengths (151) with the generating functional

$$\mathcal{Z}[\zeta,\eta] = \int \mathcal{D}\,\lambda\,\mathcal{D}\,\nu\ \exp\left[-W[\lambda,\eta] - i\,(\,\lambda\bullet\zeta + \eta\bullet\nu\,)\right]. \tag{169}$$

For this in Eq. (151) it is sufficient to represent this as

$$e^{\int_\Lambda dx\,(\,\eta B_\parallel - \frac{1}{2}\nu^2 + i\,\nu B_\parallel\,)} = e^{-\int_\Lambda dx\,\frac{\nu^2}{2}}\, e^{-\int_\Lambda dx\, i\,\eta\frac{\delta}{\delta\,\nu}}\, e^{\int_\Lambda dx\, i\,\nu B_\parallel},$$

and then functionally integrate by parts

$$\int \mathcal{D}\nu\,(\text{r.h.s. of prev. f.}) = \int \mathcal{D}\nu\, e^{-\int_\Lambda dx\,(\,\frac{1}{2}[\nu + i\,\eta]^2 - i\,\nu B_\parallel\,)},$$

and completely analogously for λ, which gives

$$Z[\zeta,\eta] = \exp\,[\,\frac{1}{2}\int_\Lambda dx\;(\,\zeta^2 + \eta^2\,)\,]\;\mathcal{Z}[\zeta,\eta]\,. \tag{170}$$

A connection formula for the averages can be deduced after performing some transformations with the shift operators in the functional space and utilising their functional integral representations. Let $F(E_\parallel, B_\parallel)$ be an arbitrary functional of the longitudinal strengths. Then the mean value

$$\langle\, F(E_\parallel, B_\parallel)\,\rangle = Z^{-1}[\zeta,\eta]\, F(\, i\,\frac{\delta}{\delta\,\zeta}\,,\,\frac{\delta}{\delta\,\eta}\,)\; Z[\zeta,\eta]\Big|_{\zeta=\eta=0} \tag{171}$$

is expressed via the mean $\langle F(\lambda,\nu)\rangle$, defining $\mathcal{Z}[\zeta,\eta]$, by means of the formula

$$\begin{array}{c}\langle\, F(\,E_\parallel\,,\,B_\parallel\,)\,\rangle = \int \mathcal{D}\,\xi\;\mathcal{D}\,\theta\;\exp\,(\,-\frac{1}{2}\int_\Lambda dx\,[\,\xi^2 + \theta^2\,]\,) \\ \langle\, F(\,\lambda + i\,\xi\,,\,-i\,\nu + \theta\,)\,\rangle\,.\end{array} \tag{172}$$

The latter relationship is the basis for the interpretation of the collective variables. They can be thought of as longitudinal (chromo) electric and magnetic fields smoothed out by a Gaussian noise. It is interesting to note, in addition, that although λ and ν are real in the integral expressions, their averages are purely imaginary

$$\begin{array}{rclcrcl}\langle\, E_\parallel\,\rangle & = & \langle\,\lambda\,\rangle\,, & & \langle\, B_\parallel\,\rangle & = & -i\,\langle\,\nu\,\rangle\,, \\ \langle\, E_\parallel\,\rangle^* & = & -\langle\, E_\parallel\,\rangle\,, & & \langle\, B_\parallel\,\rangle^* & = & \langle\, B_\parallel\,\rangle\,.\end{array} \tag{173}$$

Let us apply the stationary phase method to the action of (168). In the "classical" approximation on the variables σ and ν, which is a quasi–classical in the initial gauge fields, we need to find the minima of the effective action. We can use the natural Ansatz of constant collective variables. According to Eq. (173), we need to take the purely imaginary magnetic field at the saddle point, so we may change the notation $\nu \to i\nu$. In terms of spectral expansions for the operators ∂_t (periodic boundary conditions) and Δ_x (the Neumann condition on the boundary), we can write

$$W[\sigma,\nu] = -\frac{\beta V_R}{2}\nu^2 + 2\pi\delta(\hat{\mathbf{0}})\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\mathrm{tr}\log C_+ C_-, \tag{174}$$

$$C_\pm^{ab} = (\omega_n^2 + \Omega_m^2)\delta^{ab} + i(2\omega_n\sigma^c \pm \nu^c)gt^{abc} - (gt^{abc}\sigma^c)^2, \tag{175}$$

where $\omega_n = 2\pi\beta n$ and $\Omega_m = \pi(m+1/2)/R$. It is convenient to introduce the notations for the free energy density, $\mathcal{F}_R$,

$$W_R = \beta V_R \mathcal{F}_R\,, \quad \mathcal{F}_R = \gamma_R F_R\,, \quad \gamma_R = \frac{8\pi^2 R\delta(\hat{0})}{\beta^2 V_R}\,, \tag{176}$$

where $V_R = 4\pi R^3/3$ is the domain volume. Here $\delta(\hat{0}) = \frac{1}{4\pi}\sum_l (2l+1)$ denotes the angular delta function with coinciding arguments, which is ultraviolet divergent. It is more convenient to rewrite the result in terms of the variables $s = (g\beta/2\pi)\sigma$, $v = (g\beta/2\pi)^2\nu$.

We first restrict ourselves to the simpler case of the gauge group SU(2), and then generalise the result to SU(3) group. We denote $v = u^2$ and use the formulae

$$\prod_{m=0}^{\infty}\left(1+\frac{a^2}{(m+1/2)^2}\right) = \cosh\pi a, \quad \prod_{m=1}^{\infty}\left(1+\frac{a^2}{m^2}\right) = \frac{\sinh\pi a}{\pi a}. \tag{177}$$

It can be shown (see Appendix B), that the presence of a part of the colour vector v transverse to s leads to the imaginary contribution to the effective action. Therefore, we restrict ourselves to the Ansatz in which these vectors are collinear. Note that for the group $SU(2)$ the operator $\mathcal{L}^{ab} = t^{abc}\hat{s}^c$ has the same algebra as the operator L considered earlier in $\mathbf{x}$-space, and a determinant of the linear combination of its powers is calculated using Eq. (164) formulae. After computing the colour trace we arrive at the answer (up to an irrelevant constant)

$$F_R[u,s] = -a\,u^4 + \mathcal{U}_R[u,s]\,, \tag{178}$$

$$\mathcal{U}_R[u,s] = \mathcal{U}_R[s] + \mathcal{V}_R[u,s]\,, \tag{179}$$

$$\mathcal{V}_R[u,s] = \frac{\beta}{2\pi R}\sum_{n=-\infty}^{\infty}\log\frac{L_R((n+s)^2+u^2)\,L_R((n+s)^2-u^2)}{L_R^2((n+s)^2)}\,, \tag{180}$$

$$\mathcal{U}_R[s] = \frac{\beta}{\pi R}\sum_{m=0}^{\infty}\log\left(1-\frac{\cos 2\pi s}{\cosh(\pi(m+1/2)\beta/R)}\right)\,, \tag{181}$$

where $L_R(x) = \cosh(2\pi R\sqrt{x}/\beta)$ and $a = (2\pi)^4/(2g^2\beta^4\gamma_R)$. The expression is periodic in s and we can limit ourselves to the period $0 \le s \le 1$, or a half–period, due to parity with respect to $s = 1/2$. At finite R, calculating the derivative of the effective action on u^2 and equating it to zero, we can see that the resulting equation admits only a trivial solution $u = 0$. Further, by assuming u to be zero, it is easy to see that the minimum of the action is reached at $s = 0$. Thus, there is no phase transition in a finite size system, which can serve as an illustration of the Yang and Lee's theorem [114]. The behaviour of the effective action changes in the thermodynamic limit $R \to \infty$:

$$\mathcal{U}[s,u] \;=\; \frac{1}{2\pi^2}\int_0^\infty dv\;\log\Big[\,(\cosh\sqrt{v^2+(2\pi u)^2}-\cos[2\pi s]\,)$$

$$(\cosh \sqrt{v^2 - (2\pi u)^2} - \cos [2\pi s]) \Big/ \cosh^2 v \Big] , \tag{182}$$

$$\mathcal{U}[s,0] \;=\; -\frac{1}{2}(1-2s)^2 + \frac{1}{8} , \qquad (0 \le s \le 1) , \tag{183}$$

where we have used the formulae for calculating the integral

$$I(a,b) = \int_0^\infty dx \frac{\cosh bx}{\cosh x + a} = \frac{\pi}{\sqrt{1-a^2}} \frac{\sin(b \arccos a)}{\sin \pi b},$$

since

$$\mathcal{U}[0,s] = -2 \frac{\partial}{\partial b}\Big|_{b=1} (I(a,b) - I(0,b)) .$$

The integral can be rewritten as the series

$$\begin{aligned} \mathrm{Re}\, \mathcal{V}[s,u] \;&=\; \sum_{n=-\infty}^{\infty} \Big([\, \sqrt{(n+s)^2 + u^2} + \sqrt{(n+s)^2 - u^2}_{+} - \\ &-\quad 2 \mid n+s \mid \Big) , \\ \mathrm{Im}\, \mathcal{V}[s,u] \;&=\; \sum_{n=-\infty}^{\infty} \sqrt{u^2 - (n+s)^2}_{+} . \end{aligned} \tag{184}$$

The graphs of the real and imaginary parts are presented in Fig. 1 and 2. Applying to these series the Poisson summation formula from Appendix B we find the expansions via the cylindrical functions

$$\begin{aligned} \mathrm{Re}\, \mathcal{V}[s,u] \;&=\; 2s(s-1) + u \sum_{n=1}^{\infty} \frac{\cos[2\pi n s]}{n} \Big(Y_1(2\pi n u) - \frac{2}{\pi} K_1(2\pi n u) \Big) , \\ \mathrm{Im}\, \mathcal{V}[s,u] \;&=\; \frac{\pi u^2}{2} + u \sum_{n=1}^{\infty} \frac{\cos[2\pi n s]}{n} J_1(2\pi n u) . \end{aligned} \tag{185}$$

The asymptotic formulae can be derived from the asymptotics of the Bessel and Neumann functions and are expressed through the periodic continuation by the Hurwitz formula from Appendix B of the generalised zeta function

$$\begin{aligned} \mathrm{Re}\, \mathcal{U}[s,u] \;&=\; \sqrt{2u}\, [\, \zeta_H(-\frac{1}{2}, u+s) + \zeta_H(-\frac{1}{2}, u-s) \,] , \\ \mathrm{Im}\, \mathcal{U}[s,u] \;&=\; \frac{\pi u^2}{2} + \sqrt{2u}\, [\, \zeta_H(-\frac{1}{2}, u+s) + \zeta_H(-\frac{1}{2}, u-s) \,] . \end{aligned} \tag{186}$$

Given the periodicity and the values at the endpoints of the zeta function, it is easy to come to the following conclusion below, which is also valid beyond the limits of the applicability of the asymptotic formulae.

We calculate the derivative $\partial F[u,s]/\partial u^2$ and equate it to zero at the point $u = s = 1/2$. Hence we determine the critical the value of the parameter a

$$a_c = 2\sqrt{2} + 4\sum_{n=1}^{\infty}\left(\left((2n+1)^2+1\right)^{-1/2} - \left((2n+1)^2-1\right)^{-1/2}\right). \tag{187}$$

By using a different summation

$$(m^2 \pm 1)^{-1/2} = \frac{1}{m} + \sum_{k=0}^{\infty} \frac{(\mp 1)^k (2k-1)!!}{2^k k!} \frac{1}{m^{2k+1}} \tag{188}$$

and connection to the Riemann ζ-function $\sum_{k=0}^{\infty}(2k+1)^{-s} = (1-2^{-s})\,\zeta(s)$ we arrive at the rapidly converging series form

$$a_c = 2\sqrt{2} - 8\sum_{k=0}^{\infty} \frac{(4k+1)!!}{(2k+1)!\,2^{2k+1}} \left((1 - \frac{1}{2^{4k+3}})\,\zeta(4k+3) - 1\right), \tag{189}$$

and this can be computed numerically as approximely $a_c \simeq 2.61882$.

If the parameter "a", which has the meaning of temperature squared, is greater than the value of a_c, then there is only a trivial phase with minima at integer values of s and u. At smaller a, there are the free energy minima at half–integer values of u and s (see Fig. 3). The stability condition Im $F = 0$ (Fig. 2) gives $u \leq s$ and excludes all minima except the trivial $u = s = 0$ and non–trivial $u = s = 1/2$, the second of which is deeper. Note that the minima are also achieved at the results of the action on the $s = 1/2$ of transformations from the centre Z_2 by means of shifts $s \to s + 1$.

It is easy to see that in the thermodynamic limit, the points of the stationary phase will be not only constant fields. Indeed, the operators under the trace are diagonal over $\hat{\mathbf{x}}$, and therefore the existence of the dependence $s(\hat{\mathbf{x}})$ leads to the replacement of the value 4π by the integral over the sphere. The kinetic terms thus become nontrivial, but they are proportional to R and can be discarded in the thermodynamic limit. As we have seen, the unit vector in the colour space, to which the stationary phase solution is proportional, has been an arbitrary constant, but now we can see that it can be even an arbitrary function of $\hat{s}(\hat{\mathbf{x}})$.

Our analysis of the formulae for SU(2) group suggests, that we can restrict ourselves to an Ansatz in which the colour vectors can be simultaneously transformed to a basis with only their Cartan components ($a = 3, 8$) are different from zero, since the presence of the v part of the transverse s leads to an imaginary part of the are effective, and hence are not of interests. We introduce the notations

$$s_{\pm} \equiv \frac{1}{2}(s_3 \pm \sqrt{3}\, s_8), \qquad s_0 \equiv s_3 = s_+ + s_- \tag{190}$$

and analogously for v. Then after calculation of the colour trace one can obtain [115]

$$F_{SU(3)}[v,s] = \sum_{A=0,\pm} \left(-\frac{2a}{3} v_A^2 + \mathcal{U}_{SU(2)}[v_A, s_A] \right). \tag{191}$$

Thus, the free energy density of SU(3) group is arranged as the sum of three independent expressions of SU(2) group. By virtue of the behaviour of this function for the group $SU(2)$ one can see, then a nontrivial minimum is reached when one of the values of s_+, s_-, $s_+ + s_-$, and the other two – 1/2, and the corresponding components v are similarly zero and 1/4. Thus the minima of the effective action (see Fig. 4) are the colour vectors with components 3, 8: $\vec{s}_\pm = (1/2, \pm 1/(2\sqrt{3}))$ and $\vec{s_0} = (0, 1/\sqrt{3})$, as well as the ones obtained from them by the centre transformation actions Z_3: $s_3 \to s_3 + 1$, $s_8 \to s_8 + 1/\sqrt{3}$ or $s_3 \to s_3$, $s_8 \to s_8 + 2/\sqrt{3}$.

All these minima have the same depth, which at temperatures below critical, is greater than the depth of the trivial minima obtained by centre transformations from zero. The phase transition temperature at which the non–zero values occur, is equal to

$$T_{SU(3)} = \sqrt{\frac{3}{2}}\, T_{SU(2)}. \tag{192}$$

Allow me to return to the consideration of the dependence on the boundary variable χ. Without loss of generality, we can restrict ourselves to the formulation for SU(2) group again. Thus, the formula (168) gives us the desired representation of the form

$$Z[\chi] = \int \mathcal{D}\sigma \, \exp\left(-W[\sigma] - i \int_{\partial\Lambda} dt\, d\hat{\mathbf{x}}\, \sigma\chi \right). \tag{193}$$

In the zero stationary phase approximation, the partition function is determined by the classical equation of motion for the collective variable. We decompose the action in the neighbourhood of the saddle point

$$\begin{aligned} W[\varsigma + \sigma_1] &= W[\varsigma] + \int_\Lambda d^4x\, \frac{\delta W}{\delta \varsigma(\mathbf{x})} \sigma_1(\mathbf{x}) \\ &\quad + \int_{\partial\Lambda} dt\, R^2 d\hat{\mathbf{x}}\, \mathcal{E}^{(1)}[\varsigma]\, \sigma_1(R\hat{\mathbf{x}}) + \ldots, \end{aligned} \tag{194}$$

where $\mathcal{E}^{(1)}[\varsigma]$ is the first Euler derivative of the action. In the zero approximation, after integration over the variable at the boundary $\sigma_R(\hat{\mathbf{x}})$ we get

$$Z[\chi] \propto \exp\left(-W[\varsigma] - i \int_{\partial\Lambda} dt\, d\hat{\mathbf{x}}\, \varsigma\chi \right) \delta\left(R^2 \mathcal{E}^{(1)}[\varsigma](R\hat{\mathbf{x}}) + i\chi(\hat{\mathbf{x}}) \right), \tag{195}$$

$$\frac{\delta W}{\delta \varsigma} = 0\,. \tag{196}$$

The equation and the related boundary condition

$$\frac{\delta W}{\delta \varsigma} = 0, \qquad R^2 \varsigma'(R\hat{\mathbf{x}}) = -i\chi(\hat{\mathbf{x}}) \tag{197}$$

are rather complicated in the non–Abelian case. For small $\chi(\hat{\mathbf{x}})$ we can decompose $\varsigma = \varsigma^{(0)} + \varsigma^{(1)}$ solution near the solution with the trivial boundary condition ${\varsigma^{(0)}}'(R\hat{\mathbf{x}}) = 0$. The modulus of the solution is defined by the stationary phase value, and the unit vectors $\hat{\chi}$ and $\hat{\varsigma}$ must coincide as can be seen from the limiting behaviour at small χ.

As we have demonstrated, the Abelian theory admits only a trivial solution $\varsigma^{(0)} = 0$. In the non–Abelian theory, the same situation is realised for $T > T_c$. In this case, ς is determined from the linearised equation and the dependence $Z_R[\chi]$ has a behaviour similar to the Abelian theory. Namely, it contains a delta function expressing the the global charge conservation law, but besides this, it is trivial in the sense that $Z[\chi_{lm}] \to 1$ as $R \to \infty$.

However, at temperatures below the critical point, there is a constant non–zero solution $|\varsigma^{(0)}| = \pi/g\beta$ and the relation takes the form

$$Z[\chi] = \prod_{\hat{\mathbf{x}}} \cos \beta \varsigma^{(0)} \chi(\hat{\mathbf{x}})\,. \tag{198}$$

In this formula, we used the fact that $\varsigma^{(0)}$ can have both signs independently at every point on the sphere $\hat{\mathbf{x}}$. The product of (198) is zero if $\chi(\hat{\mathbf{x}})$ is different from zero and equal to one otherwise. Thus, only the value $\chi(\hat{\mathbf{x}}) = 0$ is statistically possible in the phase of confinement.

When calculating averages, we need to consider the measure of the phase space of the variables $\hat{\chi}$, the orbits of the adjoint transformation of the group, and either sum over all admissible values of $|\chi|$, or consider distinct "domains" with arbitrary values of this quantity. Hence, the temperature average of some observable A is given by the following formula

$$< A >= \begin{cases} \langle A \rangle_0, & T < T_c, \\ \int d\chi(\hat{\mathbf{x}})\, \langle A \rangle_\chi, & T > T_c, \end{cases} \tag{199}$$

where

$$\langle A \rangle_\chi = \frac{1}{Z[\chi]} \int \mathcal{D}\sigma\, \mathcal{D}\nu\, e^{-W[\sigma,\nu] - i\int_{\partial\Lambda} dt\, d\hat{\mathbf{x}}\, \sigma\chi}\, A[\sigma,\nu]\,. \tag{200}$$

We shall not discuss here the proper procedure for calculating averages in the deconfinement phase, although the situation seems to be analogous to the known physical systems such as spin glasses [116]. If we compute the integral over all χ at high temperature, it will yield $\delta(\sigma_\infty(\hat{\mathbf{x}}))$, and hence effectively

remove the integration over the variable at the boundary. In contrast, at low temperatures only the value $\chi = 0$ is realised and the the integral over the variable at the boundary in the mean $\langle A\rangle_0$ gives the singlet projector P_s of the group of the large gauge transformations

$$\begin{aligned}\langle A\rangle_0 &= \int \mathcal{D}\sigma_\infty(\hat{\mathbf{x}}) \lim_{R\to\infty} \frac{1}{Z_R[0]} \mathrm{Tr}\,(e^{-\beta H_R - iQ(\sigma_\infty)}\, A) \\ &= \lim_{R\to\infty} \frac{1}{Z_R[0]} \mathrm{Tr}\,(e^{-\beta H_R}\, P_s\, A)\,, \end{aligned} \tag{201}$$

where $Q(\sigma_\infty)$ are generators of these transformations.

Finally, let us discuss the symmetry with respect to the centre group Z_N (this we shall consider in more detail below), which is realised as shifts $g\beta A_0$ of a special kind. For example, for the group $SU(2)$ in the adjoint representation, only shifts to integers are allowed. Recall that the centre of the fundamental representation of $SU(N)$ is generated by the unit roots of $\exp(i2\pi n/N)$, but the adjoint representation of $SU(2)$ coincides with the adjoint representation of $SO(3)$ and so the centre elements are trivially represented in it.

From the above analysis it is obvious that there is a spontaneous breaking of Z_N symmetry at high temperature, in the deconfinement phase, by a surface term in any state with χ different from zero. In the confinement phase this symmetry is not broken, since $\chi = 0$ and the partition function contains the integration over the surface Lagrange multiplier, by virtue of which a shift of this integration variable by the value of the mean $\langle A_0\rangle$ is trivial on the surface of the delta function and does not break the symmetry. Moreover, by the action of Z_N all minima (including non–trivial ones) are transformed into each other.

6 Confinement criteria. The Wilson loop

In the considered formalism the notion of confinement has been introduced as the singletness with respect to the group of the large gauge transformations of all states contributing to the temperature averages of the observables, on which the generators of transformations are nullified,

$$\int_0^R y^2\, dy\, \Phi_\perp(y\hat{\mathbf{x}}) = 0. \tag{202}$$

Using the Gauss's law, we rewrite $\Phi_\perp = \partial \mathbf{E}_\parallel$ and integrate (202) over an arbitrary area ΔS on the boundary by converting the volume integral into a surface integral over the encompassing conic surface

plus the integral over the area itself. Obviously, the normal to the conic surface is orthogonal to $\hat{\mathbf{x}}$, and therefore the integral over it is zero. Thus, the condition (202) is equivalent to *the equality to zero of the chromo–electric field flux through any element of the boundary.* Note that the condition of the zero chromo–magnetic field flux through an arbitrary element of the boundary also naturally appeared in our approach. This formulation of confinement essentially coincides with intuitive physical notions and it leads to the unobservability of coloured states at infinity.

It would be natural to compute another physical characteristic of the colour no–escape, namely the energy $E(L)$ of a state with two static coloured sources quark – antiquark located at distance L from each other. This quantity is a function of the response of the gauge field to the introduction of sample sources and is expressed in terms of a geometric field characteristic called the Wilson loop [62, 13]

$$\mathcal{W}[\Gamma] = \langle \mathrm{Tr\,P}\, \exp(g \oint_\Gamma A_\mu dx^\mu)\rangle, \quad A_\mu = A_\mu^a t^a, \quad (t^a)^\dagger = -t^a\,, \tag{203}$$

where t^a are generators of the fundamental representation of the group $SU(N)$, and Γ is a closed contour in space–time (for definiteness we shall consider a rectangular contour of length T in the temporal direction and length L in space). The asymptotic of the Wilson loop at large T is related to the interaction energy of the charges $\mathcal{W}[\Gamma] \propto \exp(-T\,E(L))$. According to the confinement criterion by Wilson [62], the confinement phase follows the area law $\mathcal{W}[\Gamma] \propto \exp(-\chi S(\Gamma))$, and the Coulomb phase obeys the perimeter law $\mathcal{W}[\Gamma] \propto \exp(-\mu P(\Gamma))$, where $S(\Gamma) = LT$ and $P(\Gamma) = 2(T + L)$ and the area and the perimeter of the contour respectively. The proportionality parameter in the linearly rising potential energy, $E(L) = \chi\, L$, is called the string tension coefficient. Such a name has a clear visual interpretation, since the the flux of the coloured chromosome–electric field between the charges is compressed into tubes (strings) of a fixed thickness [13]. The Wilson loop is a gauge–invariant object by definition.

In lattice calculations, a different confinement criterion is more popular. It is associated with the Polyakov line [17, 71]

$$\mathcal{P}(\mathbf{x}) = \mathrm{T} \exp \int_0^\beta dt\, A_0(t, \mathbf{x}). \tag{204}$$

The temperature average of the trace of products of this operator is related to the the free energy of sample point charges introduced into the system at the points which are its arguments. The average trace of the Polyakov line, which is proportional to the partition function of the solitary charge in the field, invariant only with respect to the gauge transformations satisfying periodic boundary conditions

$U(0,\mathbf{x}) = U(\beta,\mathbf{x})$. According to another confinement criterion [17], the mean of the Polyakov line is zero in the confinement phase, i.e. the free energy of the solitary charge is infinite, and in the Coulomb phase it is different from zero. Thus, this quantity is the order parameter of the confinement-deconfinement phase transition and allows us to distinguish the global symmetry breaking with respect to the centre Z_N of the gauge group $SU(N)$. Indeed, the gauge transformations satisfying the condition $U(t+\beta,\mathbf{x}) = zU(t,\mathbf{x})$, where $z = \exp(i2\pi n/N) \in Z_N$ do not change the periodic boundary conditions on the gauge fields in the functional integral and, of course, any $SU(N)$ gauge invariant objects. In this case, they non–trivially transform the Polyakov line $\mathcal{P} \to z\,\mathcal{P}$. Therefore, in the absence of Z_N symmetry breaking (the confinement phase) $\langle\mathcal{P}\rangle = 0$, and in the deconfinement phase the mean is non–zero and the symmetry is spontaneously broken. The universality inherent to all critical [117, 118, 119] phenomena would imply that the properties of the the confinement–deconfinement phase transition for the gauge fields on a lattice should be analogous to those of any Z_N symmetric lattice system [71]. However, the continuous limit of the theory gives rise to ultraviolet and infrared singularities requiring a non–trivial renormalisation procedure. Therefore, the question of establishing a firm connection between the system on a lattice and the limiting continuous system remains a serious theoretical challenge.

The calculation of the Polyakov line by analytical methods in the continuous theory is rather difficult due to the lack of its full gauge invariance, although it is easily realisable numerically on the lattice. We shall therefore restrict ourselves to the consideration of the Wilson confinement criterion.

Let us proceed to the direct calculation of the Wilson loop in the confinement phase [120, 121]. In the FS gauge, the zero component of the of the gauge field is expressed in the form (63). For an arbitrary of the functional F it is easy to demonstrate that

$$\langle F[A_0(\mathbf{x})]\rangle = \langle \int_x^R dy\, E_\parallel(y\hat{\mathbf{x}})\rangle\,. \tag{205}$$

To obtain this ratio, we can write the mean in the form

$$F[\frac{\delta}{\delta J_0}]\Big|_0 e^{i\int_\Lambda d^4x\, A_0(\Phi - J_0)}.$$

Integrating over A_0 and expressing $E_\parallel$ using the δ-function, we indeed have

$$\begin{aligned} &F[\frac{\delta}{\delta J_0}]\Big|_0 \exp\left(\frac{1}{2}\int_\Lambda d^4x\,(E_\parallel - \frac{1}{x^2}\int_0^x y^2 dy\, J_0(y\hat{\mathbf{x}}))^2\right) \\ &= F[\int_x^R dy\, E_\parallel(y\hat{\mathbf{x}})]\,\exp\left(\frac{1}{2}\int_\Lambda d^4x\, E_\parallel^2\right)\,. \end{aligned}$$

The most convenient Γ contour consists of two straight lines along the radius in some fixed direction $\hat{\mathbf{x}}_0$ at times $t = 0, T$, which give zero contributions due to the gauge condition, and two lines parallel to the time axis at radii $x = R', R''$, for which we can write

$$\begin{aligned}\mathcal{W}[\Gamma] &= \left\langle \mathrm{Tr}\left[\left(\mathrm{T}\exp\int_0^T dt\, A_0(R''\hat{\mathbf{x}}_0)\right)^\dagger \right.\right. \\ &\left.\left. \mathrm{T}\exp\int_0^T dt\,(A_0(R''\hat{\mathbf{x}}_0) + \int_{R'}^{R''} dy\, E_\parallel(y\hat{\mathbf{x}}_0))\right]\right\rangle. \end{aligned} \tag{206}$$

For the sake of brevity, we shall further omit the angular coordinates unless this complicates the understanding of the formulae. Taking advantage of the gauge invariance of the expression under the trace, we can transform the variables as follows

$$\begin{aligned} A_0 &\to A_0' = U^{-1}(A_0 + g^{-1}\partial_t)U(t), \\ E_\parallel &\to E_\parallel' = U^{-1}E_\parallel U(t). \end{aligned} \tag{207}$$

From the definition of T-exponent $\mathcal{P}[A_0](t) = \mathrm{T}\exp\int_0^t dt'\, A_0(t')$, as the solution of the following differential equation,

$$\dot{\mathcal{P}}[A_0](t) = gA_0(t)\,\mathcal{P}[A_0](t), \quad \mathcal{P}[A_0](0) = \mathbf{1} \tag{208}$$

we obtain

$$\begin{aligned}&\mathcal{P}\left[A_0(R'') + \int_{R'}^{R''} dy\, E_\parallel(y)\right](t) \\ &= \mathcal{P}[A_0(R'')](t)\,\mathcal{P}\left[\mathcal{P}[A_0(R'')]^{-1}\int_{R'}^{R''} dy\, E_\parallel(y)\ \mathcal{P}[A_0(R'')]\right]. \end{aligned} \tag{209}$$

Under gauge transformations (40) T-exponent changes as

$$\mathcal{P}(t) \to \mathcal{P}'(t) = U^{-1}(0)\mathcal{P}(t)U(t). \tag{210}$$

To perform the Gibbs averaging, we use the formula (172), which allows us to replace the average of an arbitrary functional from $E_\parallel$ with the average of the functional from the conjugate variable λ with the help of an additional Gaussian integration over the auxiliary variable ξ. We can try to simplify the contribution of the surface term, which, as we have shown, should be taken with $\chi = 0$ in the confinement phase. The integral over $A_0(R\hat{\mathbf{x}})$ gives the surface δ-function and we can write the

partition function in the form

$$Z[0] = \int \mathcal{D}\mathbf{A}_\perp\, \mathcal{D}\mathbf{E}_\perp\, e^{-W[\mathbf{A}_\perp,\mathbf{E}_\perp]}\, \delta\left(R^2 E_\parallel(R\hat{\mathbf{x}})\right) = \int \mathcal{D}\lambda\, \mathcal{D}\mathbf{A}_\perp\, \mathcal{D}\mathbf{E}_\perp$$
$$e^{\int_\Lambda d^4x \left[\frac{1}{2}(\mathbf{E}_\perp^2 - \mathbf{B}^2) - \frac{1}{2}\lambda^2 + \lambda E_\parallel\right]}\, \delta\left(R^2 E_\parallel(R\hat{\mathbf{x}})\right)\, \delta\left(R^2\lambda(R\hat{\mathbf{x}})\right). \qquad (211)$$

Here we have used the fact that the addition of a second surface δ-function is equivalent to changing the normalisation constant due to the presence of the first δ-function. After we have taken into account the constraint on λ, we again represent the first δ-function as an integral over $A_0(R\hat{\mathbf{x}})$, integrate over the transverse physical variables and so we obtain a representation in terms of the effective action of the of collective variables of the form (152,153). Thus, we have shown that actually the integration is performed only over the fields with an additional boundary condition on $\lambda(R\hat{\mathbf{x}})$. Hence, the integration space of the conjugate variable satisfies the same boundary condition as the space $\mathbf{E}_\perp$. The Wilson loop will be expressed through the following functional integral

$$\mathcal{W}[\Gamma] = \int \mathcal{D}\xi\, \mathcal{D}\sigma\, \mathcal{D}\nu\, e^{-W[\sigma,\nu] - \frac{1}{2}\int_\Lambda d^4x\, \xi^2}\, \delta\left(R^2\lambda(R\hat{\mathbf{x}})\right)$$
$$\mathrm{tr}\, \mathcal{P}\left[\mathcal{P}[\sigma(R'')]^{-1} \int_{R'}^{R''} dy\, (\lambda + i\xi)(y)\, \mathcal{P}[\sigma(R'')]\right]. \qquad (212)$$

We have converted this formula to the Euclidean time by substituting $t \to i\,t$, and "tr" denotes the trace in the fundamental representation $SU(N)$, in contrast to "Tr" which denotes a trace in Hilbert space. Note that the coefficient functions of the fields are time dependent and hence, the T-exponent is not reducible to the ordinary exponent.

The expression (212) at finite R, R', R'' is extremely computationally difficult. One can, however, expect some noticeable simplifications in the limit of the infinite $R \to \infty$ system, when the pair quark–antiquark is located infinitely far from the centre of coordinates. In other words, we are only interested in the asymptotic interaction energy in the limit $R', R'' \to \infty,\ \Delta R = R'' - R' = const.$ We introduce the variable φ using the relation

$$\lambda(\mathbf{x}) = \frac{1}{x^2}\int_0^x y^2 dy\, \varphi(y\hat{\mathbf{x}}).$$

By virtue of the confinement condition (202), the electric field flux through an infinitely small solid angle at spatial infinity in arbitrary direction is zero. In terms of the Fourier components of the basis of the form

$$f_{lm\,n}(k)[\mathbf{x}, t] = \left(\frac{2}{\pi}\right)^{1/2} \frac{\sin kx}{kx}\, Y_{lm}(\hat{\mathbf{x}})\, \exp(i2\pi n\, t/\beta)$$

the δ-functional gives in the considered limit the condition

$$\tilde{\varphi}_{lm\,n}(k=0)=0. \tag{213}$$

The expression in the T-exponent tends to

$$\int_{R'}^{R''} dy\,\lambda_{lm\,n}(y) \to -(8\pi)^{1/2}\frac{\Delta R}{R'R''}\,\tilde{\varphi}_{lm\,n}(k=0), \tag{214}$$

and therefore can be neglected. It is easy to realise that the expression in this formula oscillates strongly and the convergence is very slow. Using such an approximation in terms of the original integral over the gauge fields would give only a trivial result, nevertheless, it is natural in the integral over the collective variables. This circumstance has the same reason as the fact of the non–triviality of the stationary phase approximation in the last representation of the of the functional integral compared to the original integral. The collective variable includes in its definition the averaging over a Gaussian noise, and therefore, in some sense its quantum fluctuations give a less significant contribution.

Once the contribution of (214) is neglected, the expression can be significantly simplified by by choosing a special gauge transformation. Let us perform some gauge transformation without changing the magnitude of the trace (212). The auxiliary variable is transformed homogeneously $\xi \to \xi' = U^{-1}\xi U(t)$. The action for ξ does not contain time derivatives, so we can replace the variables in the integral $\xi \to \xi'$ (whose Jacobian is equal to unity), without violating the periodic boundary conditions in time. Removing the prime in the notation of the integration variable, we obtain the original expression where $\mathcal{P}[A_0(R'')]$ is transformed according to the formula (210). Note also that the constant homogeneous transformation is taken out of the T-exponent and vanishes due to the cyclic nature of the trace. Thus, we have shown that the Wilson loop does not change under the transformation affecting only the construction $\mathcal{P}[A_0(R'')] \to \mathcal{P}' \equiv U^{-1}(t)\,\mathcal{P}[A_0(R'')]$.

Since $U(t)$ is arbitrary, we can get $\mathcal{P}'$ to be an arbitrary time–dependent transformation. We introduce the notation $\Xi = \int_{R''}^{R'} dy\,\xi(y)$. There holds the following

> Lemma. *For an arbitrary function of the form $\hat{\Xi}^a(t) = \Xi^a/|\Xi|$, there exists such a gauge transformation $U(t)$, which makes the corresponding element of the gauge algebra $\hat{\Xi} = \hat{\Xi}^a t^a$ independent of time.*

For prove this it suffices to show that for an arbitrary normalised vector in the colour space of the

group $SU(N)$ there always exists a (in general, not the unique) solution of equation

$$\frac{d}{dt}\left(U^{-1}(t)\,\hat{\Xi}(t)\,U(t)\right) = 0. \tag{215}$$

This equation can be rewritten as

$$[V(t)\,,\,\hat{\Xi}(t)] = \partial_t\,\hat{\Xi}(t)\,, \qquad\qquad V(t) = \partial_t\,U(t)\;U^{-1}(t)\,. \tag{216}$$

This, obviously, can be resolved in $V(t)$, and then $U(t)$ is equal to

$$U(t) = \mathrm{T}\,\exp\int^t dt'\,V(t')\,. \tag{217}$$

After choosing such a transformation, the T-exponent is reduced to the normal exponent. To get rid of the vector structure we apply an additional global transformation, so that the vector $\hat{\Xi}$ belongs to the Cartan subalgebra. Due to the diagonality of the Cartan generators, the trace, and then the Gaussian integral, are easily computed

$$\mathcal{W}[\Gamma] = \int\prod_{\alpha=1}^{N-1}\mathcal{D}\xi^\alpha\,e^{-\frac{1}{2}\int_\Lambda d^4x\,(\xi^\alpha)^2}\,\sum_{i=1}^{N}\exp\left[\sum_{\alpha=1}^{N-1}(j\bullet\xi^\alpha)\,\lambda_i^{(\alpha)}\right], \tag{218}$$

$$j(x) = \frac{i\,g}{2}\,\theta(0\le t\le T)\,\frac{\theta(R'\le x\le R'')}{x^2}\,\delta(\hat{\mathbf{x}}-\hat{\mathbf{x}}_0), \tag{219}$$

leading to the result

$$\mathcal{W}[\Gamma] = \sum_{i=1}^{N}\exp[-\sum_{\alpha=1}^{N-1}(\lambda_i^{(\alpha)})^2\,\chi_0\,S]\,, \tag{220}$$

where $\left\{\lambda_i^{(\alpha)}\;\equiv\;(\lambda^\alpha)_{i\,i}\,,\;(\alpha=1,\,\ldots,\,N-1)\right\}$ are the elements of the diagonal generators in the fundamental representation of $SU(N)$. The generator matrices are normalised as follows

$$\mathrm{tr}(\lambda^a\lambda^b) = 2\delta^{ab}\,. \tag{221}$$

The condition (221) can be formally represented in terms of N-dimensional vectors $\lambda^{(\alpha)}$, if we introduce an additional vector with the components

$$\lambda^{(0)} = \frac{1}{\sqrt{N}}\,(\underbrace{1,\ldots,1}_{N})\,, \tag{222}$$

as a condition of orthonormalisation of the N-dimensional basis $\left\{\lambda^{(0)},\lambda^{(\alpha)}/\sqrt{2}\right\}$. The completeness condition of such a basis gives

$$\sum_{\alpha=1}^{N-1}\lambda_i^{(\alpha)}\,\lambda_j^{(\alpha)} = 2\left(\,\delta_{ij}-\lambda_i^{(0)}\,\lambda_j^{(0)}\right). \tag{223}$$

The sum in (220) is easily calculated using the last relation

$$\sum_{\alpha=1}^{N-1} (\lambda_i^{(\alpha)})^2 = 2\left(1 - \frac{1}{N}\right). \tag{224}$$

Combining the previous results, we obtain the area law

$$\mathcal{W}_{SU(N)}[\Gamma] = N\, e^{-\chi_{SU(N)}\, S}\,, \tag{225}$$

with the string tension coefficient equal to

$$\chi_{SU(N)} = 2\left(1 - \frac{1}{N}\right)\chi_0\,. \tag{226}$$

The constant χ_0 contains ultraviolet and infrared divergences:

$$\chi_0 = \frac{g^2\,\delta(\hat{0})}{8\,R'\,R''}\,, \qquad \delta(\hat{0}) = \sum_l \frac{(2\,l+1)}{4\,\pi}\,, \tag{227}$$

but it has the same structure as the square of the temperature T_c of the phase transition of the confinement-deconfinement transition. The formula (227) is expressed through the bare parameters of the theory and requires the application of a renormalisation procedure, which is however difficult in non–covariant gauges.

For the group $SU(2)$ for the dimensionless combination $\xi \equiv T_c/\sqrt{\chi}$ we obtain the following formula

$$\xi^2 = \frac{a_c 8R^3}{\pi^2 V_R} = \frac{6}{\pi^2} a_c. \tag{228}$$

Given the formula (189), the numerical value is $\xi \simeq 0.71$ and is in qualitative agreement with the the result obtained from Monte Carlo simulations in the lattice theory, which equals 0.69 ± 0.02. The free energy density can now be rewritten in terms of the string tension as

$$\mathcal{F}_R[a_R] = \frac{54\,\pi}{g^2\,\beta^2}\,\chi_0\,F_R\!\left(\frac{\pi^3}{6\,\beta^2\,\chi_0}\right), \tag{229}$$

which emphasises the clearly non–perturbative character of the approximation we have considered. The computation of the Wilson loop in the deconfinement phase is much more complicated due to the absence of the δ-function, which actually has allowed us to get rid of of the most complicated functional integration above.

7 Conclusion

In conclusion, let us formulate the main results obtained in the thesis:

1. The formulation of the non–Abelian gauge theory in the 3-dimensional Fock–Schwinger gauge is constructed. A generalisation of this gauge to a class of gauges, which we call the generalised Fock–Schwinger gauges, for which the Gauss law constraint is still exactly solvable and the gauge fields are expressed via the the strength fields by a linear differential relation, is proposed.

2. The problem of the group of residual gauge transformations allowed by these gauges is studied and the choice of boundary conditions fixing such arbitrariness is discussed. Explicit formulae for the connection of gauge fields in the given gauge with those in an arbitrary gauge are obtained. These formulae are analysed in detail on the example of the Coulomb gauge in the Abelian theory, and allow one, for instance, to obtain a simple representation for the free propagator in the Fock–Schwinger gauge.

3. It is shown that among the gauges of the class considered here, there is a preferred one for the consideration of the dynamics in a given compact region, the form of which is consistent with the field of normal vectors to the boundary of the region. Such a gauge leads to a substantial simplification of the surface terms in the theory. The Hamiltonian formalism for the Yang–Mills theory in a finite domain, which allows one to take into account the variables at the boundary as true Hamiltonian variables and the the contribution of these surface terms is quite essential, whereas the formal neglect of these terms in the standard approach is invalid and leads to the loss of essential physical effects. Any boundary conditions in this formalism are thus treated as additional constraints. Admissibility of this or that boundary condition is determined by the conditions of its preservation by the dynamics of the system in the finite domain.

4. The formulation of the Yang–Mills theory in the generalised Fock–Schwinger gauge is constructed in the formalism of the "variables at infinity" proposed by Morcio and Strocchi in the framework of the algebraic Quantum Field Theory [98, 99, 100]. This formalism allows us to construct the correct limit of the dynamics of the system in infinite volume as a group of automorphisms of the extended algebra of observables with a nontrivial centre. The latter is generated by weak bounds of delocalised variables, called variables at infinity. The non–trivial dynamics of variables at infinity in the Yang–Mills theory is deduced, which is essential for the confinement mechanism. In the limit of a system of infinite size, the formalism of point 3 turns into the the present one, and thus their equiva-

lence is established. Note that the former version is useful for practical calculations, for example, of thermodynamic quantities at finite temperature and is used further in this thesis, while the latter is of great value for analysing the phenomena of spontaneous symmetry breaking, the global structure of the state space of the limiting system, and is also more convenient in the axiomatic field theory for a rigorous proof of general results. The representation of the Poincaré algebra in the Fock–Schwinger gauge is briefly discussed.

5. The main achievement of the thesis is the analysis of the dependence of the of the partition function of the non–Abelian gauge theory on the boundary value of the longitudinal component of the electric field, which by virtue of the Gauss law in the given gauge coincides with the electric field flux through an infinitesimal element of the of the boundary, and the establishment of the connection between such dependence and the mechanism of the confinement–deconfinement phase transition.

6. The analysis of the non–Abelian theory is preceded by consideration of a simpler quantum electrodynamics problem with an external charge density in a spherical region at finite temperature, where the dependence of the partition function on the boundary variable is calculated exactly. The result suggests that in the case of a charge density corresponding to a linearly increasing potential, which models the confinement situation, the surface effects lead to a finite correction that decreases the free energy density. Therefore the tendency towards confinement is a statistically favoured effect.

7. A formulation of quantum gluodynamics in terms of the functional integral of the collective variables, conjugate to the longitudinal components of the chromo–electric and –magnetic fields is developed. Remarkably, the integrals over the gauge fields themselves are exactly calculated in the Fock–Schwinger gauge. Thus, the generating functional of the theory is represented in terms of the variables transformable homogeneously under the action of gauge transformations. The effective action (free energy density) is calculated in cases of $SU(2)$ and $SU(3)$ gluodynamics in the mean–field approximation for the collective variables. The analysis of the free energy minima revealed a phase transition at a certain temperature, below which the mean value of the collective variables is different from zero.

8. It is found that the latter transition can be interpreted as a confinement–deconfinement phase transition. In the confinement phase the flux of the chromo–electric field through an arbitrary element of the boundary is strictly zero, which is the condition of singletness with respect to the group of the residual gauge transformations, and this physically implies the impossibility of observing coloured objects at spatial infinity (in asymptotic states).

9. It is shown that our confinement condition satisfies the traditional confinement criteria. Firstly, it is shown that in the confinement phase the Wilson loop for the $SU(N)$ theory satisfies the area law. The ratio of the phase transition temperature to the square root of the string tension coefficient does not contain any divergences and, despite the approximations used in the calculation of the phase transition temperature (the mean–field contribution and the lack of a full renormalisation procedure), it is in a qualitative agreement with the result obtained using the Monte Carlo simulations of the lattice gauge theory.

10. It is shown that in the deconfinement phase the global Z_N-symmetry with respect to the centre of the group $SU(N)$ is spontaneously broken by the surface terms. The confinement phase is characterised by the unbroken symmetry. In the latter phase all nontrivial minima of the effective action have the same depth and are transformed into each other by the action of Z_N transformations, which allows us to expect the structure of the ground state of the theory similar to the θ-vacuum.

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8 Appendices

Appendix A. Gluon propagator in the FS gauge

Let us consider the propagator in the Fock–Schwinger gauge

$$D^{ij}_{FS}(x,y) = -i\,\langle 0 \mid T_D\,(\,A^i_\perp(x)\,A^j_\perp(y)\,)\mid 0\rangle\,.$$

After rewriting these expressions via the Coulomb fields with the use of the algebraic properties (30,31) one can derive the following representation

$$\begin{aligned} D^{ij}_{FS}(\mathbf{x},\mathbf{y};x_0,y_0) &= -\int\frac{d^4k}{(2\pi)^4}\,\frac{e^{-i\,k_0\,(x_0-y_0)}}{(k^2+i\,0)}\Bigg[\delta^{ij}\,e^{i\,\mathbf{k}(\mathbf{x}-\mathbf{y})} - \frac{\partial}{\partial\,x_i}x^j\frac{e^{i\,\mathbf{kx}}-1}{i\,\mathbf{kx}} - \\ &- \frac{\partial}{\partial\,y_j}y^i\frac{e^{-i\,\mathbf{ky}}-1}{-i\,\mathbf{ky}} + \frac{\partial}{\partial\,x_i}x^l\frac{\partial}{\partial\,y_j}y^l\frac{e^{i\,\mathbf{kx}}-1}{i\,\mathbf{kx}}\frac{e^{-i\,\mathbf{ky}}-1}{-i\,\mathbf{ky}}\Bigg]\,. \end{aligned}$$

Due to the boundary condition for the gauge field at zero this propagator is well–defined, whereas the propagator in the axial gauge has an unpleasant double pole. Despite many similarities of our gauge with the axial one, the latter posses serious problems with the boundary conditions [122].

Appendix B. SU(2) Group: the generic Ansatz

The general formula for the free energy density in terms of the dimensionsless parameters has the form

$$F[\vec{s},\vec{v}] = -a\,\vec{v}^{\,2} + \mathcal{U}[\vec{s},\vec{v}]\,, \tag{230}$$

$$\mathcal{U}[\vec{s},\vec{v}] = \sum_n\int_0^\infty\frac{dv}{\pi}\log\det C[\vec{s},\vec{v}]\,, \tag{231}$$

$$C^{ab}[\vec{s},\vec{v}] = \delta^{ab}\,(v^2+n^2) + i\,t^{abc}\,(2ns^c+v^c) + t^{acd}t^{bed}\,s^c s^e\,. \tag{232}$$

By means of a colour transformation one achieves the following parametrisation

$$\vec{s} = (0,\,0,\,s)\,, \qquad\qquad \vec{v} = (v_\perp,\,0,\,v_\parallel)\,.$$

The determinant is then equal to

$$\det C = X^3 + 2s^2X^2 + (s^4 - \vec{v}^{\,2} - 4ns\,v_\parallel)\,X - v_\perp^2\,s^2 \tag{233}$$

with $X = v^2 + n^2$.

We denote by γ_a $(a = 1, 2, 3)$ the roots of the right–hand side of Ed. (233) on X, which will be studied below. The formula

$$\int_0^\infty \frac{dv}{\pi}\, \log(1 + \frac{a}{v^2}) = a^{1/2} \tag{234}$$

allows us to express the function (232) as

$$\mathrm{Re}\mathcal{U}[s, \vec{v}] \;=\; \sum_n [\sum_{a=1}^{3} (n^2 - \gamma_a[s, \vec{v}, n])_+^{1/2} - 3 \mid n \mid] - \frac{1}{2}\,, \tag{235}$$

$$\mathrm{Im}\mathcal{U}[s, \vec{v}] \;=\; \sum_n \sum_{a=1}^{3} (\gamma_a[s, \vec{v}, n] - n^2)_+^{1/2}\,. \tag{236}$$

Due to the periodicity and even parity $\mathcal{U}$ of s one can consider it on the half–period $0 \le s \le 1/2$. Right away we have to exclude the domain where there is a nonzero imaginary part. Obviously this is nonzero if,

$$\gamma_a[s, \vec{v}, 0] > 0\,.$$

Thus, let us find when the equation

$$x^2 + 2s^2\, x + s^4 - \vec{v}^2 = \frac{v_\perp^2\, s^2}{x} \tag{237}$$

possesses positive or complex solutions (see Fig. 5).

It is easy to see that the lower branch of the hyperbolic curve $2''$ intersects the parabola 1 in two points, giving rise to negative real roots γ_2, γ_3. It is important to emphasise the following circumstance. As soon as $v_\perp \neq 0$, the upper branch of the hyperbola $2'$ intersects the parabola in a positive point γ_1, leading to the instability. Due to this, the general case $v_\perp \neq 0$ is of no intrest.

Fig. 5 is fairly informative and even allows us to see the stable minimum $F[\vec{s}, \vec{v}]$. This function is as smaller, as greater $\mid \vec{v} \mid$ is, whike $\mid \gamma_a \mid$ — is conversely greater. The negativity requirement imposes the additional restriction $\vec{v}^2 \le s^4$, which is saturated when $\mid \vec{v} \mid = v_\parallel = s^2 = \frac{1}{4}$.

Appendix C. Some formulae with the special functions

The Poisson summation

$$\sum_{k=-\infty}^{\infty} f(2\pi k) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau\, e^{-i\,k\tau}\, f(\tau)\,.$$

Some integrals with the cylindrical functions [123]

$$\int_0^\infty dy\, \frac{\cos(ky)}{(y^2 + v^2)^{\frac{1}{2}}} = \mathrm{K}_0(kv)\,, \qquad -\int_v^\infty dy\, \frac{\cos(ky)}{(y^2 - v^2)^{\frac{1}{2}}} = \frac{\pi}{2}\mathrm{Y}_0(kv)\,,$$

$$\int_o^u v\,dv\,\mathrm{K}_0(kv) = \frac{1}{k^2}(-ku\,\mathrm{K}_1(ku)+1)\,,$$
$$\frac{\pi}{2}\int_0^u v\,dv\,\mathrm{Y}_0(kv) = \frac{1}{k^2}(\frac{\pi}{2}ku\,\mathrm{Y}_1(ku)+1)\,.$$

The Hurwitz representation of the generalise ζ-function [124] valid for $\mathrm{Re}\,s<0$

$$\zeta(s,v) = 2(2\pi)^{s-1}\,\Gamma(1-s)\sum_{k=1}^{\infty} n^{s-1}\,\sin(2\pi nv+\pi s/2)\,.$$

We use the periodic continuation of ζ-function by means of this formula and denote it as ζ_H. For any integer k

$$\zeta_H(-\frac{1}{2},k) = -\frac{\zeta(3/2)}{4\pi}\,,\qquad \zeta_H(-\frac{1}{2},k+\frac{1}{2}) = (1-1/\sqrt{2})\frac{\zeta(3/2)}{4\pi}\,.$$

References

[1] D. Cutts et al, *Phys. Rev. Lett.*, vol. 41, p. 363, 1978.

[2] L. Lyons, preprint 38/80, Oxford Univ. Nucl. Phys. Lab., 1980.

[3] V.N. Gribov, "*Orsay lectures on confinement*," preprint 92/60, 94/20, LPTHE, Orsay.

[4] C.N. Yang, R.L. Mills, *Phys. Rev.*, vol. 69, p. 191, 1954.

[5] H. Fritzsch, M. Gell-Mann, H. Leutwyler, *Phys. Lett.*, vol. B 74, p. 365, 1973.

[6] S. Weinberg, *Phys. Rev. Lett.*, vol. 31, p. 494, 1973.

[7] D.J. Gross, F. Wilczek, *Phys. Rev. Lett.*, vol. 30, p. 1343, 1973.

[8] H.D. Politzer, *Phys. Rev. Lett.*, vol. 30, p. 1346, 1973.

[9] S. Coleman, D.J. Gross, *Phys. Rev. Lett.*, vol. 31, p. 851, 1973.

[10] W. Marciano, H. Pagels, *Phys. Rep.*, vol. 36, p. 137, 1978.

[11] A.J. Buras, *Rev. Mod. Phys.*, vol. 52, p. 199, 1980.

[12] S. Mandelstam, *Phys. Rep.*, vol. 67, p. 109, 1980.

[13] M. Bander, *Phys. Rep.*, vol. 75, no. 4, p. 205, 1981.

[14] G.'t Hooft, *Nucl. Phys.*, vol. B 75, p. 461, 1974.

[15] A.M. Polyakov, *Phys. Lett.*, vol. B 59, p. 82, 1975.

[16] A.M. Polyakov, *Phys. Lett.*, vol. B 72, p. 477, 1978.

[17] A.M. Polyakov, *Gauge Fields and Strings*. Chur: Harwood Acad. Publ., 1987.

[18] I.I. Kogan, A. Kovner, "*Compact QED_3 - a simple example of a variational calculation in a gauge theory.*," preprint UMN-TH-1317-94, TPI-MINN-94/37-T, hep-th@xxx/9410067.

[19] M.G. Amaral, M.E. Pol, *J. Phys.*, vol. G 16, p. 1, 1990.

[20] C. Itzykson, J.-B. Zuber, *Quantum Field Theory*. McGraw–Hill, USA, 1980.

[21] T.-P. Cheng, L.-F. Li, *Gauge theories of elementary particles physics*. Clarendon Press, Oxford, 1984.

[22] Y. Nambu, *Phys. Rev.*, vol. D 10, p. 4262, 1974.

[23] M. Jacob, ed., *Dual theory.* North–Holland, Amsterdam, 1974.

[24] S. Mandelstam, *Phys. Rep.*, vol. C 23, p. 245, 1976.

[25] S. Mandelstam, in *In: Proc. 1979 Int. Sym. on Lepton and Photon Interactions at High Enrgies,* Fermilab, Batavia, Illinois, 1979, (H. T.B.W. Kirk, ed.).

[26] G.'t Hooft, *Nucl. Phys.*, vol. B 79, p. 276, 1976.

[27] G.'t Hooft, *Nucl. Phys.*, vol. B 153, p. 141, 1979.

[28] S. Mandelstam, *Phys. Rev.*, vol. D 19, p. 2391, 1979.

[29] S. Maedan, Y. Matsubara, T. Suzuki, *Progr. Theor. Phys.*, vol. 84, no. 1, p. 130, 1990.

[30] H. Monden et al, *Phys. Lett.*, vol. B 294, p. 100, 1992.

[31] T. Suzuki, *Phys. Rev. (Proc. Suppl.)*, vol. B 30, p. 276, 1993.

[32] G. Savvidy, *Phys. Lett.*, vol. B 71, p. 133, 1977.

[33] S.G. Matinyan, G. Savvidy, *Nucl. Phys.*, vol. B 134, p. 539, 1978.

[34] N.K. Neilson, P. Olesen, *Nucl. Phys.*, vol. B 144, p. 376, 1978.

[35] H.B. Neilson, M. Ninomiya, *Nucl. Phys.*, vol. B 156, p. 1, 1979.

[36] J. Ambjorn, N.K. Neilson, P. Olesen, *Nucl. Phys.*, vol. B 152, p. 75, 1979.

[37] H.B. Neilson, P. Olesen, *Nucl. Phys.*, vol. B 160, p. 380, 1979.

[38] Yu.A. Simonov, *Nucl. Phys.*, vol. B 307, p. 512, 1988.

[39] G.G. Callan, R.F. Dashen, D.J. Gross, *Phys. Rev.*, vol. D 17, p. 2717, 1978.

[40] G.G. Callan, R.F. Dashen, D.J. Gross, *Phys. Rev.*, vol. D 19, p. 1826, 1979.

[41] G.G. Callan, R.F. Dashen, D.J. Gross, *Phys. Rev.*, vol. D 20, p. 3279, 1979.

[42] E. Witten, *Nucl. Phys.*, vol. B 149, p. 285, 1979.

[43] S. Mandelstam, *Phys. Rev.*, vol. D 20, p. 3223, 1979.

[44] R. Anishetty, M. Baker, J.S. Ball et al, *Phys. Lett.*, vol. B 86, p. 52, 1979.

[45] J.S. Ball, F. Zach, *Phys. Lett.*, vol. B 95, p. 273, 1980.

[46] R. Oehme, *Phys. Lett.*, vol. B 232, p. 489, 1989.

[47] R. Oehme, *Phys. Rev.*, vol. D 42, p. 4209, 1990.

[48] R. Oehme, *Mod. Phys. Lett.*, vol. 8, p. 1533, 1993.

[49] A. A. Migdal, *Phys. Rep.*, vol. 102, no. 4, p. 199, 1983.

[50] Yu. Makeenko, A.A. Migdal, *Phys. Lett.*, vol. B 88, p. 135, 1979.

[51] J.L. Gervais, A. Neveu, *Phys. Lett.*, vol. B 80, p. 255, 1979.

[52] G. S. Irosnikov, *JETP* (Journal of Experimental and Theoretical Physics, Russia), vol. 97, p. 424, 1990.

[53] G. S. Irosnikov, *JETP* (Journal of Experimental and Theoretical Physics, Russia), vol. 100, p. 45, 1991.

[54] G.'t Hooft, *Nucl. Phys.*, vol. B 72, p. 461, 1974.

[55] E. Witten, *Nucl. Phys.*, vol. B 160, p. 57, 1979.

[56] T. Kugo, I. Ojima, *Progr. Theor. Phys.*, vol. 66, p. 1, 1979.

[57] T. Kugo, I. Ojima, *Progr. Theor. Phys.*, vol. 60, no. 6, p. 1869, 1978.

[58] T. Kugo, I. Ojima, *Progr. Theor. Phys.*, vol. 61, no. 2, p. 644, 1979.

[59] H. Hata, I. Niigata, *Nucl. Phys.*, vol. B 389, p. 133, 1993. hep-th/9405145.

[60] M. Shintani, *Phys. Lett.*, vol. B 137, p. 220, 1984.

[61] T. Kugo, I. Ojima, *Progr. Theor. Phys.*, vol. 61, p. 294, 1979.

[62] K. Wilson, *Phys. Rev.*, vol. D 10, p. 2445, 1974.

[63] J. Kogut, L. Susskind, *Phys. Rev.*, vol. D 11, p. 395, 1975.

[64] L. Susskind, *Phys. Rev.*, vol. D 20, p. 2610, 1979.

[65] M. Creutz, *Phys. Rev.*, vol. D 21, no. 8, p. 2308, 1980.

[66] L. McLerran, B. Svetitsky, *Phys. Lett.*, vol. B 98, p. 195, 1981.

[67] K. Kuti, J. Polonyi, K. Szlachanyi, *Phys. Lett.*, vol. B 98, p. 199, 1981.

[68] J. Engels, F. Karsch, H. Satz et al, *Phys. Lett.*, vol. B 101, p. 89, 1981.

[69] J. Engels, F. Karsch, H. Satz, *Nucl. Phys.*, vol. B 205, p. 545, 1982.

[70] E. Marinari, *Phys. Rep.*, vol. 184, no. 2-4, p. 131, 1989.

[71] B. Svetitsky, *Phys. Rep*, vol. 132, no. 1, p. 1, 1986.

[72] B. Svetitsky, L. Yaffe, *Nucl. Phys.*, vol. B 210, no. [FS6], p. 423, 1982.

[73] B. Svetitsky, L. Yaffe, *Phys. Rev.*, vol. D 26, p. 963, 1982.

[74] S. Jacobs, M. G. Olson, C. Suchyta III, *Phys. Rev.*, vol. D 33, p. 3338, 1986.

[75] E. Marinari, *Nucl. Phys. (Proc. Suppl.)*, vol. B, no. 9, p. 209, 1989.

[76] F. Karsch, "*Deconfinement and chiral symmetry restoration on the lattice*," in *QCD 20 Years Later*, 9-13 Jun 1992, Aachen, (P.M. Zerwas, H.A. Kastrup, ed.), pp. 717–747, World Scientific, Singapore, 1993.

[77] I.I. Kogan, A. Kovner, "*A variational approach to the QCD wave functional: Dynamical mass generation and confinement.*," preprints LA-UR-94-2727, PUPT-1492, hep-th@xxx/9408081.

[78] N. A. Sveshnikov, 1995 (unpublished).

[79] A. Khvedelidze, V. Pervushin, "*Zero mode of Gauss constraint in gaugeless reduction of Yang–Mills theory*," preprint JINR E2-94-332.

[80] F.A. Lunev, *Phys. Lett.*, vol. B 295, p. 99, 1992.

[81] M. Lavelle, D. McMullan, "*On quark confinement*," preprints MZ-TH/93-03, DIAS-STP-93-04.

[82] K.J. Dahlem, *Z. Phys.*, vol. C 29, p. 553, 1985.

[83] J.E. Mandula, *Phys. Lett.*, vol. B 210, p. 117, 1988.

[84] R. Anishetti, *J. Phys.*, vol. G 10, p. 423, 1984.

[85] V.M. Belyaev, V.L. Eletzky, *Z. Phys.*, vol. C 45, p. 355, 1990.

[86] K. Enqvist, K. Kajantie, *Z. Phys.*, vol. C 47, p. 291, 1990.

[87] J. Polonyi, S. Vazques, *Phys. Lett.*, vol. B 240, p. 183, 1990.

[88] V.M. Belyaev, *Phys. Lett.*, vol. B 254, p. 153, 1991.

[89] V. V. Skalozub, preprint IP-92-12E, Institute of Theoretical Physics, Kiev, 1992.

[90] V. A. Fock, *Sow. Phys*, vol. 12, no. HEFT 4, p. 404, 1937.

[91] J. Schwinger, *Phys. Rev.*, vol. 82, p. 664, 1951.

[92] L. Durand, E. Mendel, *Phys. Rev.*, vol. D 26, p. 1368, 1982.

[93] M. Shifman, *Nucl. Phys.*, vol. B 173, p. 13, 1980.

[94] D. Shütte, *Phys. Rev.*, vol. D 40, p. 2090, 1989.

[95] N.A. Sveshnikov, E.G. Timoshenko, *Phys. Lett.*, vol. B 289, p. 423, 1992.

[96] N.A. Sveshnikov, E.G. Timoshenko, "*Confinement phase transition mechanism of SU(2)-gluodynamics*," preprint 91-140, IHEP, Serpukhov, 1991.

[97] E.G. Timoshenko, "*Dynamics of gauge fields in finite domain,*" in *VIII Workshop on High Energy Physics and QFT*, 17-23 Sep 1993, Zvenigorod, (B.B. Levchenko, ed.), (Moscow), pp. 213–217, NPI MSU, MSU Publishing Co., 1994.

[98] G. Morchio, F. Strocchi, *Commun. Math. Phys.*, vol. 99, p. 153, 1985.

[99] G. Morchio, F. Strocchi, *J. Math. Phys.*, vol. 28, no. 8, p. 1912, 1987.

[100] G. Morchio, F. Strocchi, *J. Math. Phys*, vol. 28, no. 3, p. 622, 1987.

[101] V. E. Zakharov, *Zh. Prikl. Mekh. Tekhn. Fiz.* (Russia) vol. 2, p. 86, 1968.

[102] V.S. Buslaev, L.D. Faddeev, L.A. Takhtajan, *Phys.*, vol. D 18, p. 255, 1986.

[103] D. Lewis, J. Marsden, R. Montgomery and T. Ratiu, *Physica*, vol. D 18, p. 391, 1986.

[104] T. Regge, C. Teitelboim, *Ann. Phys. (N.Y.)*, vol. 88, p. 286, 1974.

[105] V.O. Soloviev, *Phys. Lett.*, vol. B 292, p. 30, 1992.

[106] A.P. Balachandran, G. Bimonte, K.S. Gupta, A. Stern, *Int. J. Mod. Phys.*, vol. A 7, p. 4655, 1992.

[107] V.O. Soloviev, *J. Math. Phys.*, vol. 34, p. 5747, 1993.

[108] A. A. Kirillov, *Elements of the theory of representations.* NY: Springer, 1976.

[109] G.G. Emch, *Algebraic methods in Statistical Mechanics and Quantum Field Theory.* Wiley–Interscience, 1972.

[110] O. Brately, D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics.* Vol. 1,2, N. Y.: Springer Verlag, 1979, 1981.

[111] P. Bona, *J. Math. Phys.*, vol. 30, no. 12, p. 2994, 1989.

[112] P. Besting, D. Schütte, *Phys. Rev.*, vol. D 42, p. 594, 1990.

[113] N.A. Sveshnikov, E. G. Timoshenko, "Boundary effects in the gauge field theories at finite temperature," 1995. *Int. J. Mod. Phys.*, (to be published).
Consequently appeared in: *Phys. Rev.* vol. D 58, No 8, pp. 085024 -085024-9 (1998).

[114] J. Glimm, A. Jaffe, *Quantum Physics. A functional Integral Point of View.* N. Y.: Springer–Verlag, 1981.

[115] N.A. Sveshnikov, E.G. Timoshenko, "*Confinement phase transition mechanism in gluodynamics,*" preprint 92-31, IHEP, Serpukhov, 1992.

[116] M. Mezard, G. Parisi, M.A. Virasoro, *Spin Glass Theory and Beyond*. Singapore: World Scientific, 1987.

[117] J.J. Binney, N.J. Dowrick, A.J. Fisher, M.E. Newman, *The theory of critical phenomena*. Clarendon Press, Oxford, 1992.

[118] C. Itzykson, J.-M. Drouffe, *Statistical Field Theory*. Cambridge Univ. Press, 1989.

[119] D.J. Amit, *Field Theory, Renormalisation Group and Critical Phenomena*. World Scientific, Singapore, 1978.

[120] E.G. Timoshenko. "*Algebraic confinement and Wilson loop in SU(N) gluodynamics.*" *Phys. Atom. Nucl.* [Journal of Nuclear Physics (Russia), pp. 277-284], vol. 56, No 11, pp. 1613-1616, 1993.

[121] E.G. Timoshenko, "*Wilson Loop for SU(N)-gluodynamics*," preprint 92-17/266, NPI MSU, Moscow, 1992.

[122] A. Basseto, I. Lazzizera, R. Soldati, *Nucl. Phys.*, vol. B 236, p. 319, 1984.

[123] A.P. Prudnikov, Yu. A. Brychkov, O.I. Marichev, *Integrals and series*. Transl. by N.M. Queen. N. Y.: Gordon and Breach Science Publishers,1986.

[124] H. Bateman, A. Erdelyi, *Higher Transcedental Functions*. Vol. 1, MC Graw–Hill, 1953.

[125] N.A. Sveshnikov, E.G. Timoshenko, *Problems on High Energy Physics and Field Theory* – Proceedings of the XV Workshop, IHEP, Protvino, 1995, No. 162.

[126] E.G. Timoshenko, *Problems on High Energy Physics and Field Theory* – Proceedings of the XV Workshop, IHEP, Protvino, 1995, No. 169.

Figure Captions

Fig. 1. Dependence of the real part of the effective potential on u and s at $a = 0.01$.

Fig. 2. Dependence of the imaginary part of the effective potential on u and s.

Fig. 3. Plot of the dependence of Re F on u at fixed s at the phase transition ($s = 0$, $a = 0.38$, $s = 0.5$, $a = 2$).

Fig. 4. Minima of the effective action in $SU(3)$ gluodynamics in terms of the variable $s = g\beta\sigma/(2\pi)$.

Fig. 5. Plots for determining the roots γ_i of the equation $\det C(n = 0) = 0$ for group $SU(2)$. Parabola 1 crosses x axis in points $-s^2 \pm |\vec{v}|$.

Figures

Fig. 1

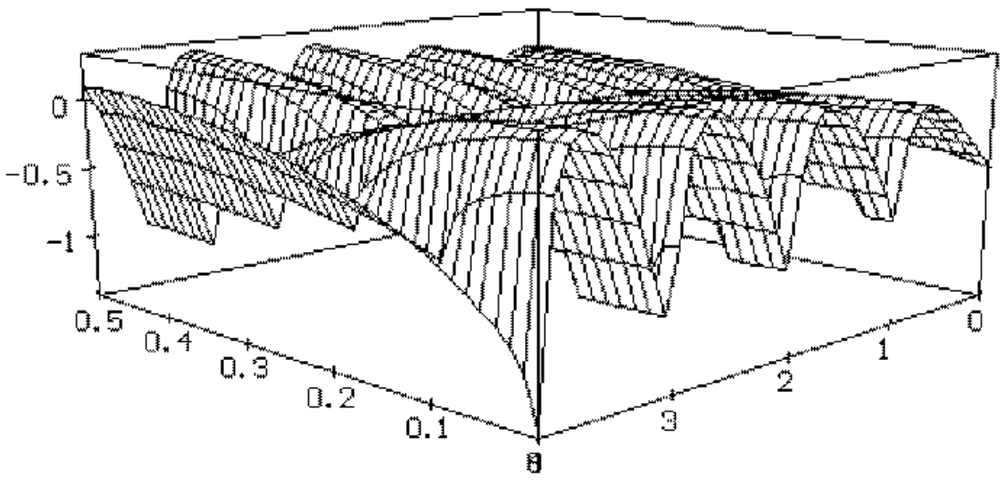


Fig. 2

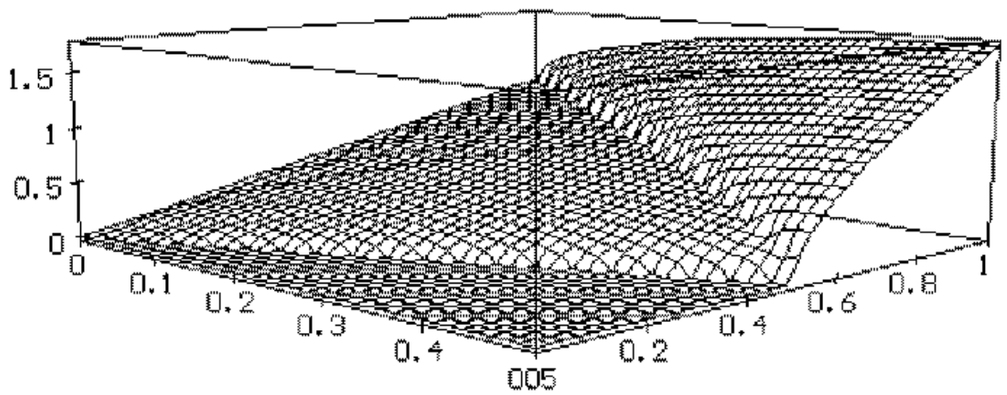
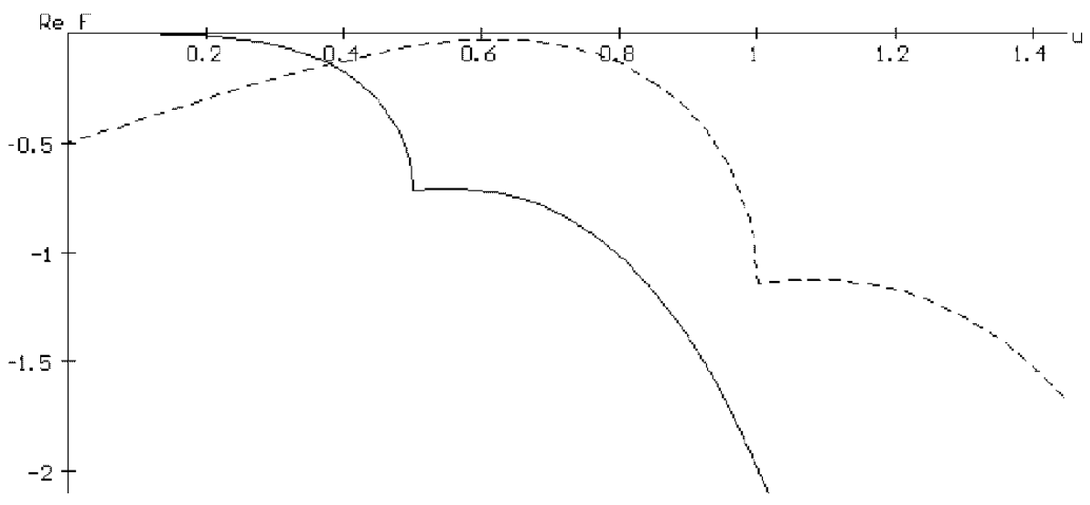


Fig. 3



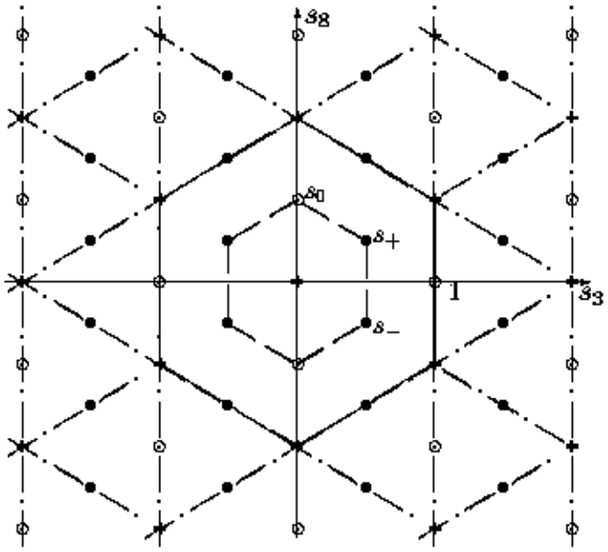


Fig. 4

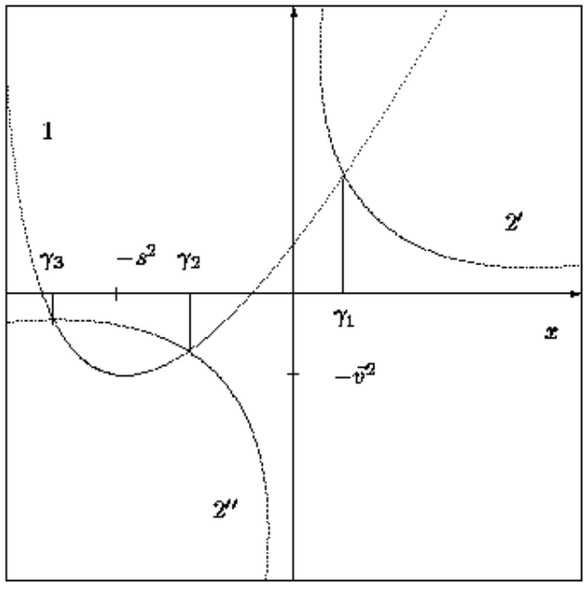


Fig. 5

EXTRACT

from the decision of the Special Council K.053.05.24
of M.V. Lomonosov Moscow State University
the 27th of April 1995

By the decision of the Special Council **K.053.05.24** of Moscow M.V. Lomonosov State University (record of proceedings No 3 of the 27th of April 1995) **Edward Georgievich Timoshenko** has been conferred the degree *candidate of physical-mathematical sciences* after defence of the thesis ``Boundary Effects and Confinement in the Theory of Nonabelian Gauge Fields'' on the speciality 01.04.02 theoretical physics.

The thesis has been prepared in the Nuclear Physics Institute of Moscow State University.

Scientific adviser: Dr. N.A. Sveshnikov.

Official referees: Prof. A.S. Vshivtsev and Dr. F.V. Tkachov gave positive reports on the thesis.

Leading institution: Institute for High Energy Physics (Protvino) gave a positive report on the thesis.

The defendant has 7 published scientific papers on the subject including the following:

- N.A. Sveshnikov, E.G. Timoshenko, *Phys. Lett.* **B 289**, 423 (1992).
- E.G. Timoshenko, *J. Nucl. Phys.* **56 (11)**, 277 (1993).

Note: the degree *candidate of physical-mathematical sciences* corresponds to the degree Doctor of Philosophy (*Ph.D.*)

Chairman of the Council
Corresponding Member of the Russian Academy of Sciences

Professor G.B. Khristiansen

Scientific Secretary of the Council