DYNAMICS OF GAUGE FIELDS IN FINITE DOMAIN

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Hamiltonian formalism for Yang–Mills system enclosed into a finite domain is constructed without discarding surface terms in equations of motion. Boundary conditions on canonical variables are treated in this formalism like usual constraints. It is shown that localised evolution of gauge fields is recovered when values of field strengths on the boundary are fixed on some values. Presented analysis is of importance for describing global properties of nonabelian gauge theory and in particular confinement problem.

1. Introduction

Severe infrared divergences in Yang–Mills theory pose real obstacle for obtaining reliable information on global properties of the system. Confinement phenomenon [1] is still one of the incompletely understood phenomena at certain extent due to this problem. Inevitable shortcomings of infrared regularization therefore require some care. Of various methods available we prefer the oldest and the most physical one, namely the system enclosed into a finite domain.

Traditional approach is then to choose some boundary conditions on canonical variables thereby ensuring unique solution of differential equations on classical level. However the problem gets rather more complicated when equations are nonlinear and elliptic type constraints are present in the theory. Really, values inside and on the boundary are no longer independent as follows from the Gauss law constraint.

We aim here at developing consistent approach when boundary conditions are considered alongside and on equal footing with any other constraints. In other words we analyse dependence of Yang–Mills theory on possible boundary conditions. This more general treatment oblige us to avoid unproved assumptions that some surface terms are vanishing. Proceeding with a finite–volume Hamiltonian in this scheme one arrives at nonlocal equations of motion generally. Later restricting a bit arbitrariness on the boundary one can recover locality on some constraints.

Recently V.O. Soloviev [2] has suggested a modified Poisson bracket for a class of field theoretic models inspired by study of surface waves in hydrodynamics [3]. That bracket differs from the standard one by some surface integrals and satisfy Jacobi identity strictly, but not up to some divergences as standard does. We give here a particular case of that general construction sufficient for our modest purposes.

Let us consider a class of local functionals of the form

$$F = \int_V d\mathbf{x}\; f[\varphi^A;\, \varphi^A_{,i}] \tag{1}$$

depending on not higher than first derivatives of canonical variables φ^A and possessing by definition the canonical Poisson brackets

$$\{\varphi^A(\mathbf{x}), \varphi^B(\mathbf{x}')\} = I^{AB}\, \delta(\mathbf{x}, \mathbf{x}')\,. \tag{2}$$

Here I^{AB} is a usual simplectic matrix with the properties: $I^2 = -1,\; I = const$. Then a generic variation of functional F with respect to canonical variables

$$\delta F = \int_V d\mathbf{x} \left(\mathcal{E}_A(F)\, \delta\varphi^A + \partial_i\big(\mathcal{E}^i_A(F)\, \delta\varphi^A\big) \right) \quad (3)$$

written in above form defines so–called Eulerian derivatives of zero and first orders:

$$\mathcal{E}_A(F) = \frac{\partial f}{\partial \varphi^A} - \partial_i \frac{\partial f}{\partial \varphi^A_{,i}}\,, \quad \mathcal{E}^i_A(F) = \frac{\partial f}{\partial \varphi^A_{,i}}. \quad (4)$$

In these terms time evolution and Poisson bracket of two such local functionals are defined as follows

$$\dot F = \{F, H\} = \int_V d\mathbf{x} \Big(\mathcal{E}_A(F)\, I^{AB}\, \mathcal{E}_B(H) \quad (5)$$
$$+\partial_i\big(\mathcal{E}^i_A(F)\, I^{AB}\, \mathcal{E}_B(H) + \mathcal{E}_A(F)\, I^{AB}\, \mathcal{E}^i_B(H)\big)$$
$$+\partial_i\partial_j\big(\mathcal{E}^i_A(F)\, I^{AB}\, \mathcal{E}^j_B(H)\big)\Big).$$

The main result of the paper [2] was in that these Poisson brackets satisfy all necessary properties as antisymmetry, completeness and Jacobi identity. It is interesting to note that the same equations of motion can be derived directly from the action principle without discarding any surface terms:

$$S = \int_V d\mathbf{x} \left(\frac{1}{2} \dot\varphi^A\, I^{-1}_{AB}\, \varphi^B - h(\varphi^A; \varphi^A_{,i}) \right), \quad (6)$$
$$\delta S = \int_V d\mathbf{x} \Big(\dot\varphi^A\, I^{-1}_{AB}\, \delta\varphi^B - \mathcal{E}_B(H)\delta\varphi^B$$
$$- \partial_i\big(\mathcal{E}^i_B(H)\delta\varphi^B\big)\Big).$$

Explicit form of domain V is clearly irrelevant provided it is closed and smooth. However for technical reasons we construct domain V as follows. Let $\{X_k; k = 1,2,3\}$ be orthonormal curvilinear coordinates system in $\mathbf{R}^3$. Boundary manifold ∂V is defined by the equation $\partial V = \{X_k : X_1 = R = const\}$, parameter R is called infrared regulator. If $\{x_k; k = 1,2,3\}$ denote usual Cartesian coordinates in $\mathbf{R}^3$, the local frame of the curvilinear coordinates may be presented in the form

$$e^{(k)}_i = h_k^{-1} \frac{\partial x_i}{\partial X_k}\,, \quad h_k = \left(\sum_{i=1}^{3} \left(\frac{\partial x_i}{\partial X_k} \right)^2 \right)^{1/2}. \quad (7)$$

We do not distinguish super- and subscripts. Summation over repeated indexes is always assumed unless otherwise noted. Vector components in the local frame will be denoted by parentheses

$$A_{(k)} = e^i_{(k)}\, A_i, \quad \partial_{(k)} \equiv h_k^{-1} \frac{\partial}{\partial X_k}. \quad (8)$$

Vector $\mathbf{e}_{(1)}$ is obviously normal to the boundary ∂V. It is natural to introduce $(2+1)$ decomposition onto longitudinal and transversal (greek) components : $i \;\to\; (1, \alpha)$, $\alpha = 2, 3$. Fixing the gauge where longitudinal component vanishes one will be able to decouple dynamics of variables on the boundary.

2. Equations of motion

Let us proceed from naive Yang-Mills Hamiltonian in finite volume

$$\mathcal{H}_V = \int_V d\mathbf{x} \Big(\frac{1}{2} (E^a_i)^2 + \frac{1}{4} (F^a_{ij})^2 \quad (9)$$
$$- A^a_0 \left(\partial_i E^a_i - g t^{abc} A^b_i E^c_i \right) \Big).$$

It was first realised by T. Regge and C. Teitelboim [4] that such Hamiltonian should be improved using integration by parts in the Gauss term. They argued this would be necessary for the Hamiltonian to generate any local equations of motion at all. In our more refined analysis we are not restricted by similar motives just now, so new Hamiltonian is simply postulated. Moreover to study dependence of the theory on boundary conditions let us introduce also arbitrary function on the boundary $\chi = \chi(X_2, X_3)$ into the surface term

$$\Delta H_V = \int_{\partial V} dX_2\, dX_3\; A_0^a \left(\frac{h}{h_1} E^a_{(1)} - \chi^a \right), \quad (10)$$

where $h \equiv h_1 h_2 h_3$. Hamiltonian $H_V = \mathcal{H}_V + \Delta H_V$ and usual simplectic matrix with $I_{A_i^a E_j^b} = \delta^{ab}\delta_{ij}$ lead to the following equations of motion

$$\int_V d\mathbf{x}\, g_i^a \dot{A}_i^a = \int_V d\mathbf{x}\, g_i^a (E_i^a + \nabla_i A_0^a), \quad (11)$$

$$\int_V d\mathbf{x}\, g_i^a \dot{E}_i^a = \int_V d\mathbf{x}\, g_i^a (-g\, t^{abc} E_i^b A_0^c + \nabla_j F_{ij}^a) - \int_{\partial V} dX_2 dX_3\, g^a_{(\alpha)}\; \frac{h}{h_1} \left(\frac{1}{h_{(\alpha)}} \partial_{(1)} (h_\alpha A^a_{(\alpha)}) - \frac{1}{h_{(1)}} \partial_{(\alpha)} (h_1 A^a_{(1)}) \right). \quad (12)$$

Integration with a smooth test function $g_i^a(\mathbf{x})$ could not be removed since only distributions have certain dynamics. Varying with respect to the momenta conjugated to A_0 yields the Gauss law constraint

$$0 = \int_V d\mathbf{x}\, g^a\, \nabla_i E_i^a - \int_{\partial V} dX_2 dX_3\, g^a \left(\frac{h}{h_1} E^a_{(1)} - \chi^a \right). \quad (13)$$

By independence of test functions inside and on the boundary it actually breaks down into two constraints:

$$\nabla_i E_i^a = 0, \quad (14)$$

$$\left. \frac{h}{h_1} E^a_{(1)} \right|_{\partial V} = \chi^a(X_2, X_3). \quad (15)$$

The second constraint is something new and does not come up in formal treatment.

To examine equations (11,12) further it is useful to choose the so-called coordinate gauge condition

$$A^a_{(1)}(\mathbf{x}, t) = 0 \quad (16)$$

that is generalisation of the 3-d Fock–Schwinger gauge [5,6]. In this gauge the Gauss law could be solved explicitly

$$E^a_{(1)} = -\frac{h_1}{h} \int_{X_1^{(0)}}^{X_1} dX_1' \,(h \Phi_\perp)(X_1', X_2, X_3), \quad (17)$$

where $\Phi_\perp \equiv \nabla \mathbf{E}_\perp$ and $X_1^{(0)}$ is an unknown constant. In complete analogy one can solve the identity

$$\nabla_i G_i^a = 0, \quad G_i^a \equiv \nabla_j F_{ij}^a \quad (18)$$

expressing $G^a_{(1)}$ via transversal physical components. The following relation between gauge field and strengths also holds

$$\frac{1}{h_\alpha} \frac{\partial}{\partial X_1} \left(h_\alpha A^a_{(\alpha)} \right) = h_1 F^a_{(\alpha)(1)}. \quad (19)$$

Finally the Bianchi identity

$$\epsilon_{ijk}\, e^i_{(\alpha)} \nabla_j E_k^a = 0, \quad (\alpha = 2, 3) \quad (20)$$

is rewritable like this

$$\frac{1}{h_\alpha} \frac{\partial}{\partial X_1} \left(h_\alpha E^a_{(\alpha)} \right) = \nabla_{(\alpha)} \left(h_1 E^a_{(1)} \right). \quad (21)$$

Later on let us consider e.g. the case of spherical domain and the Fock–Schwinger gauge correspondingly. Denoting

$$X_1 = |\mathbf{x}|, \quad \hat{\mathbf{x}} = \mathbf{x}/X_1 \quad (22)$$

$$Q^a(X_1) = X_1^2 E^a_{(1)}(X_1 \hat{\mathbf{x}}), \quad (23)$$

$$\mathcal{F}^a_{(\alpha)}(X_1) = \frac{\partial}{\partial X_1} \left(h_\alpha A^a_{(\alpha)} \right) (X_1 \hat{\mathbf{x}}) \quad (24)$$

we propose to choose in equations (17,18) the boundary conditions

$$Q^a(0) = 0, \quad \mathcal{F}^a_{(\alpha)}(0) = 0. \quad (25)$$

They should be stable under the time evolution to have any sense. Since (11) does not contain surface term one may work with it just like with any local equation. So we calculate $\dot{\mathcal{F}}^a_{(\alpha)}$ from it by diffirentiation over X_1 and using identity (21). Likewise from (12) it is clear that $\dot{E}^a_{(1)}$ also satisfies a local equation. Therefore we get the evolution of new variables in the form

$$\dot{\mathcal{F}}_{(\alpha)} = [A_0, \mathcal{F}_{(\alpha)}], \tag{26}$$

$$\dot{Q} = [A_0, Q] - \frac{X_1}{h} \frac{\partial}{\partial X_\alpha} \left(\frac{h}{h_\alpha} \mathcal{F}_{(\alpha)} \right) + [X_1 A_{(\alpha)}, \mathcal{F}_{(\alpha)}] \tag{27}$$

with $E^a_{(1)} = -\frac{\partial A^a_0}{\partial X_1}$. Square brackets stand for usual commutators in the adjoint representation of $SU(N)$. Taking $X_1 = 0$ one can see that constraints (25) are conserved in time. Equation (15) requires that $Q^a(R) = \chi^a$. So at $X_1 = R$ the conserved constraints are only

$$Q^a(R) = \chi^a, \quad \mathcal{F}^a_{(\alpha)}(R) = 0. \tag{28}$$

Any nontrivial $\mathcal{F}^a_{(\alpha)}(R)$ leads to that "boundary" dynamics does not close containing limiting value of yet another variable $A_{(\alpha)}$ and so forth.

3. Reduced dynamics

We have proved that the constraints written now for arbitrary domain and its gauge

$$\left.\frac{h}{h_1} E^a_{(1)}\right|_{\partial V} = \chi^a, \qquad \left.\frac{h}{h_1} F^a_{(1)(\alpha)}\right|_{\partial V} = 0, \tag{29}$$

$$\left.\frac{h}{h_1} E^a_{(1)}\right|_{X_1 = X_1^0} = 0, \quad \left.\frac{h}{h_1} F^a_{(1)(\alpha)}\right|_{X_1 = X_1^0} = 0 \tag{30}$$

are conserved in time. They are of the first order. Here X_1^0 denotes centre of the coordinate system. Dynamics (11,12) on these constraints gets effectively localised. Namely, equations of motion for physical variables on the constraints are

$$\dot{A}^a_{(\alpha)} = E^a_{(\alpha)} + \nabla_{(\alpha)} A^a_0, \tag{31}$$

$$\dot{E}^a_{(\alpha)} = -g\, t^{abc} E^b_{(\alpha)} A^c_0 - \nabla_{(i)} F^a_{(i)(\alpha)}, \tag{32}$$

$$A^a_0 = \int_{X_1}^{R} dX_1' \, (h_1 E^a_{(1)})(X_1', X_2, X_3), \tag{33}$$

where $E^a_{(1)}$ is given by (17). Reduced Hamiltonian $\tilde{\mathcal{H}}_V$ obtained as the first piece of (9) in terms of physical variables commutes with the generator of large gauge transformations

$$Q_V(g) = \int_{\partial V} dX_2 dX_3 \, g^a(X_2, X_3) \, \frac{h}{h_1} E^a_{(1)}. \tag{34}$$

Therefore we deal with residual gauge symmetry of the system still present in the theory. There is no way to fix it naturally. *Ad hoc* boundary conditions seem to us unacceptable. This symmetry has certainly topological origin and is quite physical.

4. Partition function

All considerations so far were quite classical. It is tempting to ask about what is going on in the quantum theory? In the rest of the report we consider application of this formalism for evaluation of partition function. For all states give their contribution to it, partition function investigation is very useful in analysis of global properties of the whole space of states. In terms of physical variables in FS gauge it may be presented as follows

$$Z_{\beta\, V}[\chi] = \mathrm{Tr}_{\mathcal{G}_\chi} \exp\left(-\beta \tilde{\mathcal{H}}_V\right) \tag{35}$$

$$= \int \mathcal{D}A_{(\alpha)}\, \mathcal{D}E_{(\alpha)} \;\; e^{-\int_0^\beta dt\, \tilde{\mathcal{H}}_V} \, \delta(Q(R) - \chi). \tag{36}$$

Here boundary conditions (29,30) are assumed. Now expressing delta–function via integral over functions on the sphere $\sigma = \sigma(\hat{\mathbf{x}}, t)$

$$\delta(Q(R) - \chi) = \int \mathcal{D}\sigma \, e^{i \int_0^\beta dt \int d\hat{\mathbf{x}}\, \sigma (Q(R) - \chi)} \tag{37}$$

and performing integrations over canonical variables one gets the presentation in terms of effective action $W_{\beta\, V}[\sigma]$

$$Z_{\beta\, V}[\chi] = \int \mathcal{D}\sigma \, e^{-W_{\beta\, V}[\sigma]} \, e^{-i \int_0^\beta dt \int d\hat{\mathbf{x}}\, \sigma \chi}. \tag{38}$$

Surely one might as well study the effective action to get information about the partition function. The latter approach has certain advantages.

In the limit $V \to \infty$ partition function diverges such that $\lim_{V\to\infty} \log Z_{\beta V}/V$ tends to a finite limit. At the same time $W_{\beta\, V}[\sigma]$ has less singular dependence on its argument. From the estimate $Z_{\beta\, V}[\chi] \le Z_{\beta\, V}[0]$ it follows that there is only two possibilities: either the partition function has maximum at zero or it does not depend on χ at all. More precisely, these two cases may be realised differently for χ_{00} and $\chi_{lm},\;\; l,m \neq 0$ coefficients in decomposition over spherical functions. In free QED $\lim_{V\to\infty} Z_{\beta\, V}[\chi_{00}] \propto \delta(\chi_{00})$ and is *const* for any other coefficients. It is natural to expect in the nonabelian theory that there exists confinement when similar dependence holds for all l,m. In the stationary phase approximation this behaviour is determined by stationary points of the effective action.

The effective action was calculated in our previous papers [7,8] using the saddle point method. It was discovered that at some temperature its nontrivial minima $\sigma \neq 0$ arise. The latter correspond to maximum partition function at strictly zero χ. In other words at this phase values of electric field strength on the boundary are zero. Formally one has

$$Z_{\beta\, V}[0] = \int \mathcal{D}\sigma\, e^{-W_{\beta\, V}[\sigma]} = \mathrm{Tr} e^{-\beta \tilde{\mathcal{H}}_V}\, \mathcal{P}_s, \quad (39)$$

$$\mathcal{P}_s = \int \mathcal{D}\sigma\, e^{i\, Q_V(\sigma)}. \quad (40)$$

So singlet projector of the group of gauge transformations at infinity $\mathcal{P}_s$ arrises inside the trace. Consequently such phase might be interpreted as *confined* as long as any observable's average contains singlet contributions only. Wilson loop average calculated with the singlet projector [9] exhibits area law what is usually regarded as standard confinement criterion.

On the contrary, at higher temperatures the effective action possesses trivial minimum only. This would mean that $Z_{\beta\, V}[\chi]$ does not depend on χ. Therefore at the *deconfined* phase any boundary conditions for the electric field are equivalent. Thus one should integrate over all χ values as well while calculating observables' averages. The latter integral effectively removes integration over σ putting it to zero.

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