Confinement phase transition in gluodynamics

N. A. Sveshnikov, E. G. Timoshenko

Institute for Nuclear Physics, Moscow State University
119899, Moscow, Russia[1]

The general mechanism of gluon confinement phase transition via variables at infinity is clarified in 3-d Fock–Schwinger gauge at finite temperature. The obtained values of critical temperature and string tension are in agreement with lattice data.

[1]E-mail: boos@compnet.npimsu.msk.su

Confinement property seems to be the most intriguing puzzle of Yang–Mills theory for years. Many attempts were made to achieve comprehension of underlain phenomena [1], yet with mild success. The reason is mostly that perturbation expansions fail to catch truly collective effects far beyond their scope of validity. In this letter we suggest a nonperturbative approach plausibly capable to shed light on the confinement mechanism.

In nonperturbative treatment removal of superfluous (unphysical) degrees of freedom would be highly desirable. Therefore one has to fix a suitable physical gauge to work in. As such we choose the 3-dimensional Fock–Schwinger gauge [2-4]. (The covariant version of this gauge gained wide popularity in QCD applications; see e. g. [5, 6] and ref. therein.)

$$\hat{\mathbf{x}}\,\mathbf{A}(t,\mathbf{x}) = 0\,, \qquad \hat{\mathbf{x}} = \frac{\mathbf{x}}{x}\,, \quad x = |\,\mathbf{x}\,|\,. \tag{1}$$

In the FS gauge, the $\mathbf{x}$-transversal components of electric field strengths $\mathbf{E}_\perp$ and gauge fields $\mathbf{A}_\perp$ are the canonical variables of YM theory obeying the CCR's

$$[\mathbf{A}_\perp(\mathbf{x}),\, \mathbf{E}_\perp(\mathbf{y})] = i\,(\mathbf{1} - \hat{\mathbf{x}}\otimes\hat{\mathbf{x}})\,\delta(\mathbf{x}-\mathbf{y})\,. \tag{2}$$

Moreover the Gauss law constraint $\nabla\mathbf{E} = 0$ possesses the explicit simple solution

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_\perp + \hat{\mathbf{x}}\,E_\parallel\,, \\ E_\parallel(\mathbf{x}) &= -\frac{1}{x^2}\int_0^x y^2\,dy\,\nabla\mathbf{E}_\perp(y\hat{\mathbf{x}})\,, \end{aligned} \tag{3}$$

that being plugged into the YM Hamiltonian yields no more than quartic expression versus canonical variables. The price for these simplifications is the lack of manifest Poincaré covariance, and nonlocality of the Hamiltonian.

Let us consider the YM partition function Z at inverse temperature β. It may be written as path integral over fields with periodic boundary conditions in time

$$\begin{aligned} Z = \int \mathcal{D}\mathbf{A}_\perp\,\mathcal{D}\mathbf{E}_\perp\,\exp\Big[\int_0^\beta dt \int d\mathbf{x}\, f_R^2(\mathbf{x}) \\ \big(i\,\dot{\mathbf{A}}_\perp\mathbf{E}_\perp - \frac{1}{2}\mathbf{E}^2 - \frac{1}{2}\mathbf{B}^2\big)\Big]\,, \end{aligned} \tag{4}$$

where $\mathbf{B}$ denotes the magnetic field strength. The cut–off function f_R with the properties

$$f_R(\mathbf{x}) = \begin{cases} 1, & |\,\mathbf{x}\,| \le R' \qquad R' = (1-\varepsilon)R \\ 0, & |\,\mathbf{x}\,| \ge R'' \qquad R'' = (1+\varepsilon)R\,. \end{cases} \tag{5}$$

was introduced in (4) to prevent infrared singularities. To proceed further we exploit the simple observation that only the longitudinal parts of field strengths contain nonlocal as well as nonabelian terms. Namely we make the integrand of (4) gaussian in $\mathbf{A}_\perp$ and $\mathbf{E}_\perp$ by introducing additional "scalar" fields via

$$e^{-\frac{1}{2}\int dx\, f_R^2\, E_\parallel^2} = \int \mathcal{D}\lambda\, e^{\int dx\,[-\frac{1}{2}\lambda^2 + i\, f_R\, \lambda\, E_\parallel]}, \tag{6}$$

$$e^{-\frac{1}{2}\int dx\, f_R^2\, B_\parallel^2} = \int \mathcal{D}\nu\, e^{\int dx\,[-\frac{1}{2}\nu^2 + i\, f_R\, \nu\, B_\parallel]}. \tag{7}$$

The only nonlocal term is now

$$\int d\mathbf{x}\, f_R(x)\, \lambda(\mathbf{x})\, E_\parallel(\mathbf{x})\,, \tag{8}$$

which may be rewritten through

$$\int d\mathbf{x}\, \nabla \mathbf{E}_\perp(\mathbf{x})(f_R(x)\sigma(\mathbf{x}) + \sigma_R(\mathbf{x}))\,. \tag{9}$$

The new variables

$$\begin{aligned} \sigma(\mathbf{x}) &= \int_x^\infty dy\, \lambda(y\hat{\mathbf{x}})\,, \\ \sigma_R(\mathbf{x}) &= \int_x^\infty dy\, f_R'(y)\, \sigma(y\hat{\mathbf{x}}) \end{aligned} \tag{10}$$

effectively localize the theory. Note that $\sigma_R(\mathbf{x})$ gets purely delocalized after the infrared cut–off removing ($R \to \infty$) because supp $f_R'(x) = [R', R'']$. Accordingly it fails to possess a well–defined limit unless special conditions on the space of states have been imposed. There lies a deep algebraic theory behind it developed mainly by G. Morchio and F. Strocchi [7-9]. From this point of view σ_R tends to what is called a *variable at infinity*[2]

$$\sigma_R(\mathbf{x}) \overset{R\to\infty}{\longrightarrow} \sigma_\infty(\hat{\mathbf{x}})\,. \tag{11}$$

Roughly speaking the latter is a "classical" variable labelling different representations of the observables algebra. Later on we shall return to discussion of this important point. At present stage σ_∞ may be looked at as some kind of order parameter appearing in the YM theory, and our next task is to find out it's equilibrium value.

Performing integrations over $\mathbf{A}_\perp$, $\mathbf{E}_\perp$ in (4) (see [10] for more details) we get

$$\begin{aligned} Z[\beta, \sigma_\infty] &= \int \mathcal{D}\,\sigma\, \mathcal{D}\,\nu\, \exp\left(-W[\sigma + \sigma_\infty, \nu]\right), \\ W[\sigma, \nu] &= \frac{1}{2}\nu^2 - \frac{1}{2}\sigma\,\Delta\,\sigma + \frac{1}{2}K_- \bullet C_+^{-1} \bullet K_+ + \\ &\quad + \frac{1}{2}K_+ \bullet C_-^{-1} \bullet K_- + \frac{1}{2}\mathrm{tr}\,\log\, C_+ C_-\,. \end{aligned} \tag{12}$$

[2] σ_∞ is a kind of weak limit in the algebraic QFT.

$$C_\pm = -\Delta_x - \nabla_t^2 \pm D\,, \quad K_\pm = \partial_\pm \nu \pm \nabla_t\,\partial_\pm \sigma\,, \tag{13}$$

$$\nabla_t^{ab} = \delta^{ab}\,\partial_t - gt^{abc}\,\sigma^c\,, \qquad D^{ab} = gt^{abc}\,\nu^c\,. \tag{14}$$

In (12) the dot denotes integration over t and $\mathbf{x}$ together with summation over space and colour indices, Δ_x is the radial part of the Laplacian. Projected derivatives are defined by means of

$$\partial_\pm^i = \Pi_\pm^{ij}\,\partial_j\,, \tag{15}$$

$$\Pi_\pm^{ij} = \frac{1}{2}(\delta^{ij} - \hat{x}^i\,\hat{x}^j \pm i\,\varepsilon^{ijk}\,\hat{x}^k)\,. \tag{16}$$

Certainly, the most plane thing we could do with the effective action $W[\sigma,\nu]$ was to implement the saddle point method. In this letter we consider only the "classical" level leaving discussion of "quantum" corrections to further publications. So we are looking for minima of the effective action $W[\sigma,\nu]$. The simplest possible *Ansatz* of constant σ and ν is used that is, however, rather natural from physical viewpoint. Denoting[3] $v^a = i\frac{\beta g}{2\pi}\nu^a$, $s^a = \frac{\beta g}{2\pi}\sigma^a$ we present the effective potential (= free energy density) in the form

$$\mathcal{F} = \frac{8\pi^2}{\beta^2\,g^2}\,\chi_0\,F[s^a, v^a]\,, \tag{17}$$

clearly emphasizing its nonperturbative character. Here

$$F[s^a, v^a] = -a(v^a)^2 + \mathcal{U}[s^a, v^a]\,,$$

$$\mathcal{U}[s^a, v^a] = \pi^{-1}\sum_{n=-\infty}^{\infty}\int_0^\infty dq\,\log\,\det\mathcal{C}\,, \tag{18}$$

$$\mathcal{C}^{ab}[s^a, v^a] = \delta^{ab}(q^2+n^2) + i\,t^{abc}(2n\,s^c + v^c) + t^{abc}\,t^{bcd}\,s^c s^e \tag{19}$$

with

$$a = \frac{\pi^2}{\beta^2\,\chi_0}\,. \tag{20}$$

In (17) and (20), χ_0 is a certain combination of infrared and ultraviolet regulators with dimension of $(mass)^2$. The consistent theory would somehow relate it to confinement radius and UV normalization point, but this theory is still far beyond of our present knowledge. Nevertheless it may be shown [11, 12] that χ_0 is nothing but the string tension of SU(2) gluodynamics!

[3] both s^a and v^a are real

For the SU(2) group, explicit analysis [10] of function $\mathcal{U}[s^a, v^a]$ shows that the vectors $s^a,\ v^a$ must be collinear, otherwise the imaginary part of F is nonvanishing. Then $\mathcal{U}$ is periodic versus s, and for $0 \le s \le 1$

$$\mathcal{U}[s,v] = -\frac{1}{2}(1-2s)^2 + \sum_{k=-\infty}^{\infty}\Bigg(\sqrt{(k+s)^2+v} + \sqrt{(k+s)^2-v} - 2\mid k+s\mid\Bigg)\,. \tag{21}$$

The real part of F plotted on Fig. 1 is a rather complicated function. The available positions of its minima are $s=0,\ u=n$ and $s=\frac{1}{2},\ u=n+\frac{1}{2}$ with $n\in\mathbf{Z}$, and $u=\sqrt{\mid v\mid}$. In any case, however, just two of them with $n=0$ are stable belonging to the region where $\operatorname{Im} F = 0$ (Fig. 2). The second minimum is the deeper one, but it emerges only below the critical temperature

$$T_c = \sqrt{a_c\,\chi_0}/\pi\,, \tag{22}$$

$$a_c = 2\sqrt{2} - 8\sum_{k=0}^{\infty}\frac{(4k+1)!!}{(2k+1)!\,2^{2k+1}} \left[(1-\frac{1}{2^{4k+3}})\,\zeta(4k+3) - 1\right] \simeq 2.61882\ldots \tag{23}$$

The instant of the phase transition is depicted on Fig. 3.

For SU(3) group the situation is quite similar. It is easy to show that under the same assumption one has [11]

$$\mathcal{U}[s^a, v^a] = \mathcal{U}[s_3, v_3] + \sum_{\pm}\mathcal{U}[(s_3 \pm \sqrt{3}\,s_8)/2, (v_3 \pm \sqrt{3}\,v_8)/2]\,, \tag{24}$$

and consequently

$$T_{SU(3)} = \sqrt{3/2}\;T_{SU(2)}\,. \tag{25}$$

Thus we have found out the phase transition (presumably of the first order) giving rise to nonzero value of σ_∞, namely

$$\sigma_\infty^0 = \frac{\pi}{\beta\,g}\,. \tag{26}$$

The essence of this phase transition can be clarified by the help of long–range dynamics formalism by G. Morchio and F. Strocchi [7-9]. Roughly speaking, this elaborated algebraic theory considers models in which quantum equations contain delocalized variables commuting with all local ones. *Variables at infinity* are mathematically rigorous counterparts of this theory. They have definite c-number values in any primar representation. Quantum dynamics is unitarily implementable only in irreducible (vacuum) representations for which the corresponding values of variables at infinity satisfy certain triviality condition. Symmetry transformations enlarge every representation of the kind via including elements of nontrivial centre generated by variables at infinity. The (unique) extension of time evolution to the enlarged algebra gives rise to nontrivial *dynamics at infinity*. Then the carrier space stable under both time evolution and symmetry transformations is a direct integral of primar representation spaces.

Since there is a direct relation between time evolution of a system and its equilibrium state, i. e. via the KMS condition [13, 14], we have to take dynamics at infinity into account while constructing the Gibbs state. Namely, the correct averaging procedure should include integration over variables at infinity with Gibbs factor $e^{-\beta h}$, where h is the Hamiltonian of dynamics at infinity, as well as summation over different vacuum sectors. Proceeding in this way we get eventually the equilibrium state satisfying the KMS condition and invariant under symmetry transformations.

In [10] we have shown that this general scheme is applicable to the YM theory in FS gauge. In a primar representation specified by particular values of variables at infinity $\sigma_\infty(\hat{\mathbf{x}})$ the partition function (4) is

$$Z[\beta,\sigma_\infty] = \mathrm{Tr}\, e^{-\beta\,(H_{YM}+i\,\Phi[\sigma_\infty])}\,, \tag{27}$$

where

$$\Phi[\sigma_\infty] = \int d\hat{\mathbf{x}}\,\sigma^a_\infty(\hat{\mathbf{x}})\lim_{R\to\infty} R^2\,E^a_\parallel(R\hat{\mathbf{x}}) \tag{28}$$

with $E_\parallel$ given by the Eq. (3). The triviality conditions may be written as

$$Z[\beta,\sigma_\infty] = Z[\beta,0] \tag{29}$$

Then the σ_∞'s corresponding to vacuum representations are

$$\sigma^a_\infty(\hat{\mathbf{x}}) \equiv \sigma^a_n, \quad \mid \sigma_n \mid = \frac{2\pi\, n}{\beta\, g}, \;\; n \in \mathbf{Z}^+\,. \tag{30}$$

The dynamics at infinity depends on σ_n and its Hamiltonian is defined via Kirillov canonical structure [15].

The symmetry transformations form the group of *gauge transformations at infinity*

$$G_\infty = G/G_{pr}\,, \tag{31}$$

where G is the whole gauge group and G_{pr} the subgroup of proper (small) gauge transformations (for $g \in G_{pr}\; g(\mathbf{x}) \stackrel{|\mathbf{x}|\to\infty}{\longrightarrow} 1$). In other words G_∞ contains gauge transformations depending on $\hat{\mathbf{x}}$ only. These transformations act nontrivially on variables at infinity, and the $\Phi[\sigma_\infty]$ are their generators.

It may be shown that summation over vacuum representations (30) obliterates specific features of dynamics at infinity, and the correct partition function takes the form

$$Z = \int d\sigma_\infty\, Z[\beta,\sigma_\infty]\,. \tag{32}$$

ratio	lattice	theoretical
$\frac{T_c^{SU(3)}}{T_c^{SU(2)}}$	1.16 ± 0.07	$\sqrt{3/2} \simeq 1.22$
$\frac{\xi^{SU(3)}}{\xi^{SU(2)}}$	1.04 ± 0.18	$3/(2\sqrt{2}) \simeq 1.06$

Table 1: Comparison of our theoretical predictions with lattice data.

We would like to stress that this representation for Z is a direct consequence of **nontrivial** dynamics of variables at infinity in contrast to the standard picture of spontaneous symmetry breaking. In (32), $d\sigma_\infty \equiv \prod_{\hat{\mathbf{x}}} d\sigma_\infty(\hat{\mathbf{x}})$ is the Haar measure of G_∞ group, and integration is to be performed over a convex region including the equilibrium value of σ_∞. Due to (26) this is the whole group range below critical temperature. Taking (27) into account we see that

$$Z = \begin{cases} \mathrm{Tr}\, e^{-\beta\, H_{YM}}\,, & T > T_c \\ \mathrm{Tr}\,(e^{-\beta\, H_{YM}}\, P_s), & T < T_c\,, \end{cases} \tag{33}$$

where P_s is the *local singlet projector at infinity* by virtue of the well–known Peter–Weyl theorem [16]. The same projector P_s appears inside any localized variables' correlation functions below T_c. Its emergence prevents localized colour objects from propagating to spatial infinity leading to physically acceptable picture of colour confinement. Presence of P_s was shown by one of the authors (E. T. [12]) to ensure the area law for Wilson loop in fundamental representation of SU(N)

$$\mathcal{W}_{SU(N)} = N\, e^{-2(1-\frac{1}{N})\,\chi_0\, S} \tag{34}$$

with the same χ_0 as in (23). There is a surprising agreement between our formulae and results of lattice simulations. Namely, for the dimensionless ratio $\xi \equiv \frac{T_c}{\sqrt{\chi}}$ we predict in the case of SU(3) group the value

$$\xi_{SU(3)} = \frac{3}{2\sqrt{2}\,\pi}\sqrt{a_c} \approx 0.55\,, \tag{35}$$

whereas the lattice datum [17-21] is 0.58 ± 0.04. Another data are summarized in Tab. 1.

The chief issue of our investigations is thus the following. On quantum level the Yang–Mills theory accepts natural extension via incorporating extra classical degrees of freedom. The latter are conjugate to generators of gauge transformations at infinity, in the sense that they contribute to the partition function as in (27). At temperatures below T_c the equilibrium values of these quantities are nonzero. This fact entails an abrupt change of the equilibrium state structure at T_c providing colour confinement.

It should be stressed that these statements are in fact gauge independent. The specific rôle played by the Fock–Schwinger gauge is that the necessity of incorporating variables at infinity manifests itself most clearly. Having established this fact we are able to use any gauge for practical calculations. Namely, it is easy to show in general case

$$Z[\beta] = \int_{G_\infty} d\zeta \int \mathcal{D}A \, \mathrm{Det} \left[\frac{\delta \, \chi(A^\omega)}{\delta \, \omega} \right]_{\omega=0} \delta[\chi(A)] \\ \exp \left\{ -W_{YM}[A_0 + (g\beta)^{-1} \, \zeta, \mathbf{A}] \right\} \tag{36}$$

where $\chi[A] = 0$ is the gauge condition. In (36) the path integral over A is evidently the partition function in presence of the external field

$$A_0^{ext} = (g\beta)^{-1} \, \zeta(\hat{\mathbf{x}}) \, . \tag{37}$$

It is worth to note also that integration over ζ restores gauge- and Poincaré-invariance of Z. As for the $T = 0$ limit, it is permitted in formulae like (36) only *after* the ζ-integration. One of reasons for this is that correct description of long–range dynamics needs infrared regulators, and nonzero temperature is one of them.

We entertain a hope that the use of (36) may facilitate the calculation of condensates, correlation functions etc., and the analysis of renormalization procedure, that is in any case indispensable for complete substantiation of our approach. However these points are not clear yet.

We are grateful to E. E. Boos for his permanent support of this research and useful discussions. One of us (E. T.) would like to thank IHEP (Serpukhov), where a part of this work was done, for hospitality and to A. N. Solomin for assistance in LaTeX typewriting.

References

[1] M. Bander. *Phys. Rep.* **75 (4)** (1981) 205

[2] V. A. Fock. *Sow. Phys* **12 (HEFT 4)** (1937) 404

[3] J. Schwinger. *Phys. Rev.* **82** (1951) 664

[4] D. Shütte. *Phys. Rev.* **D 40** (1989) 2090

[5] L. Durand, E. Mendel. *Phys. Rev.* **D 26** (1982) 1368

[6] M. Shifman. *Nucl. Phys.* **B 173** (1980) 13

[7] G. Morchio, F. Strocchi. *J. Math. Phys* **28 (3)** (1987) 622

[8] G. Morchio, F. Strocchi. *Commun. Math. Phys.* **99** (1985) 153

[9] G. Morchio F. Strocchi. *J. Math. Phys.* **28 (8)** (1987) 1912

[10] N. A. Sveshnikov, E. G. Timoshenko. Preprint 91-140, IHEP, Serpukhov, 1991

[11] N. A. Sveshnikov, E. G. Timoshenko. Preprint 92-31, IHEP, Serpukhov, 1992

[12] E. G. Timoshenko. Wilson loop for SU(N) group. Preprint 92-17/266, NPI MSU, Moscow, 1992

[13] G. G. Emch. *Algebraic methods in Statistical Mechanics and Quantum Field Theory.* Wiley–Interscience, 1972

[14] O. Brattely, D. W. Robinson. *Operator Algebras and Quantum Statistical Mechanics*, v 1,2. Springer, N. Y., 1979, 1981

[15] A. A. Kirillov. *Elements of the theory of representations.* Springer, NY, 1976

[16] A. O. Barut, R. Rączka. *Theory of Group Representations and Applications*, v. 1. PWN, Warszawa, 1977

[17] E. Marinari. *Nucl. Phys.* (Proc. Suppl.) **B (9)** (1989) 209

[18] S. Jacobs, M. G. Olson, and C. Suchyta III. *Phys. Rev.* **D 33** (1986) 3338

[19] J. Engels, F. Karsch, and H. Satz. *Nucl. Phys.* **B 205** (1982) 545

[20] M. Creutz. *Phys. Rev. Lett.* **45 (5)** (1980) 313

[21] M. Creutz. *Phys. Rev.* **D 21 (8)** (1980) 2308